PEAK-CONSTRAINED DESIGN OF NONRECURSIVE DIGITAL FILTERS WITH LOW PASSBAND/STOPBAND ENERGY RATIO

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Abstract: *A new approach for designing nonrecursive digital filters is proposed. The method is able to compromise the minimax efficiency of maximizing the stopband attenuation and the ability of the weighted least-squares (WLS) method to minimize the total stopband energy. The approach uses a time-varying weight function which is made constant, at each iteration, inside a given frequency interval. For that matter, a partially WLS-like and partially equiripple stopband is achieved along with an equiripple passband. Efficient implementations of the new method present computational complexity comparable to minimax-based approaches. Examples are included demonstrating the good results achieved with the proposed scheme.*

1. INTRODUCTION

The design of finite-duration impulse response digital filters is dominated in the literature by the Chebyshev and the weighted least-squares (WLS) approaches. The Chebyshev scheme minimizes the maximum absolute value of a weighted error function between the prototype's transfer function and a given ideal solution. For that reason, Chebyshev filters are also said to satisfy a minimax criterion. The WLS approach, which minimizes the sum of the squares of the same weighted error function as the minimax approach, is characterized by a very simple implementation. Its basic problem, however, results from the well-known Gibbs phenomenon which corresponds to large error near discontinuities of the desired response.

The universal availability of minimax computer routines has motivated its spread use in many problems where it is not the most appropriate solution. In fact, some applications that use narrow-band filters, like frequency division multiplexing for communications, do require both the maximum stopband attenuation and the total stopband energy to be considered simultaneously. For these cases, Adams has shown in [1] that both the minimax and the WLS approaches are unsuitable as they completely disregard one of these two measurements when designing nonrecursive digital filters.

For that matter, we propose a new approach for designing peak-constrained nonrecursive digital filters with low stopband energy. The organization of this paper is as follows. In the next section, the general problem of designing linear-phase nonrecursive digital filters is presented. In Section 3, the classical optimization methods for solving the approximation problem of nonrecursive digital filters are described. These include the minimax and the WLS approaches, the Lawson algorithm [2], and the so-called Lim-Lee-Chen-Yang (LLCY) algorithm [3]. The last two are seen as methods that implement the minimax approach through a series of WLS designs. In Section 4, a new method is given based on a simple modification of the Lawson or LLCY algorithms, resulting in an excellent compromise of all good properties of the minimax and WLS methods.

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2. PROBLEM FORMULATION

Consider a nonrecursive filter of length N described by the transfer function

$$\tilde{H}(z)=\sum_{n=0}^{(N-1)} h(nT)z^{-n} \tag{1}$$

and assume that N is odd[1], $h(n)$ is symmetrical[1], and $\omega_s = 2\pi$, such that $T = 1$. The frequency response of such filter is then given by

$$\tilde{H}(e^{j\omega})=e^{-j\frac{(N-1)}{2}\omega}\hat{H}(\omega) \tag{2}$$

where

$$\hat{H}(\omega)=\sum_{n=0}^{c} a_n\cos(n\omega) \tag{3}$$

with $c=\frac{(N-1)}{2}$, $a_0 = h(c)$, and $a_n = 2h(c-n)$, for $n = 1,\dots,c$. If $e^{-jc\omega}H(\omega)$ is the desired frequency response and $W(\omega)$ is a strictly positive weighting function, consider the weighted error function $E(\omega)$ defined in the frequency domain as

$$E(\omega)=W(\omega)[H(\omega)-\hat{H}(\omega)] \tag{4}$$

The approximation problem for linear-phase nonrecursive digital filters resumes to the minimization of some objective function of $E(\omega)$ in such way that $|E(\omega)|\leq\delta$, and then

$$|H(\omega)-\hat{H}(\omega)|\leq\frac{\delta}{W(\omega)} \tag{5}$$

Evaluating the weighted error function on a dense frequency grid with $0\leq\omega_i\leq\pi$, for $i=1,\dots,MN$, a good discrete approximation of $E(\omega)$ can be obtained. Points associated to the transition band are disregarded, and the remaining frequencies should be linearly redistributed in the passband and stopband to include their corresponding edges. Thus, the following vector equation holds

$$\mathbf{e}=\mathbf{W}\left(\mathbf{h}-\mathbf{U}\mathbf{a}\right) \tag{6}$$

[1] Other cases of N even and $h(n)$ antisymmetrical can be dealt with in a very similar way [4] and are not further discussed in this paper.

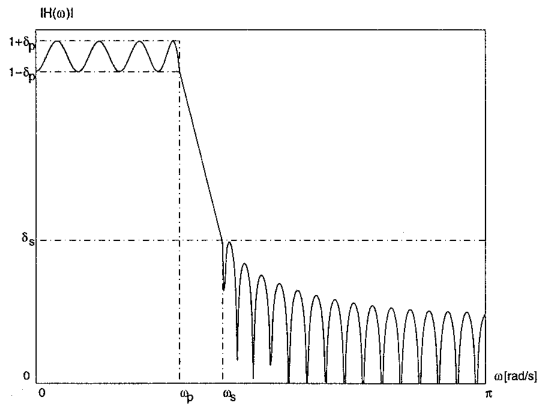

Figure 1. Lowpass filter specifications.

where

$$\mathbf{e} = [E(\omega_1)\ E(\omega_2)\ \ldots\ E(\omega_{\bar{M}N})]^T \quad (7a)$$
$$\mathbf{W} = \text{diag}\,[W(\omega_1)\ W(\omega_2)\ \ldots\ W(\omega_{\bar{M}N})] \quad (7b)$$
$$\mathbf{h} = [H(\omega_1)\ H(\omega_2)\ \ldots\ H(\omega_{\bar{M}N})]^T \quad (7c)$$
$$\mathbf{U} = \begin{bmatrix} 1 & \cos(\omega_1) & \cos(2\omega_1) & \ldots & \cos(c\omega_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \cos(\omega_{\bar{M}N}) & \cos(2\omega_{\bar{M}N}) & \ldots & \cos(c\omega_{\bar{M}N}) \end{bmatrix} \quad (7d)$$
$$\mathbf{a} = [a_0\ a_1\ \ldots\ a_c]^T \quad (7e)$$

with $\bar{M} < M$, as the original frequencies in the transition band were discarded.

An ideal lowpass filter is represented in Figure 1, where δ_p is the passband maximum ripple, δ_s is the stopband minimum attenuation, and ω_p and ω_s are the passband and stopband edges, respectively. Based on these values, define

$$DB_p = 20\log_{10}\left(\frac{1+\delta_p}{1-\delta_p}\right)\ \text{dB} \quad (8a)$$
$$DB_s = 20\log_{10}(\delta_s)\ \text{dB} \quad (8b)$$

The design of a lowpass digital filter as specified in Figure 1, using either the minimax method or the WLS approach, is achieved making the ideal response and weight functions respectively equal to [4]

$$H(\omega) = \begin{cases} 1, & \text{for } 0 \le \omega \le \omega_p \\ 0, & \text{for } \omega_s \le \omega \le \pi \end{cases} \quad (9)$$

$$W(\omega) = \begin{cases} 1, & \text{for } 0 \le \omega \le \omega_p \\ \delta_p/\delta_s, & \text{for } \omega_s \le \omega \le \pi \end{cases} \quad (10)$$

3. CLASSICAL OPTIMIZATION METHODS

3.1. Minimax Approach

Minimax design consists on the minimization over the set of filter coefficients of the maximum value of $|E(\omega)|$, i.e.,

$$\| E(\omega) \|_\infty = \min_{\mathbf{a}} \max_{0\le\omega\le\pi} [W(\omega)|H(\omega) - \hat{H}(\omega)|] \quad (11)$$

With the discrete set of frequencies, using equation (7), the minimax function becomes

$$\| E(\omega) \|_\infty \approx \min_{\mathbf{a}} \max_{0\le\omega_i\le\pi} [\mathbf{W}|\mathbf{h} - \mathbf{U}\mathbf{a}|] \quad (12)$$

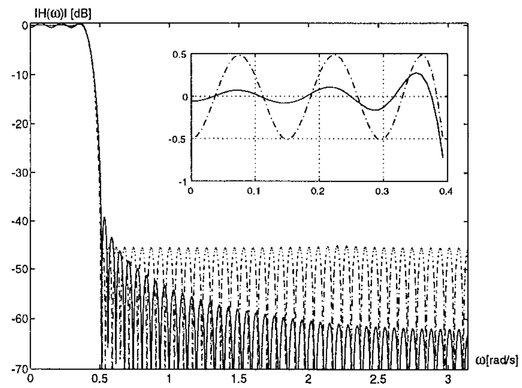

Figure 2. Lowpass frequency response of minimax- (dash-dot) and WLS-based (solid) filters.

Referring to Figure 1, the minimax method optimizes

$$DB_\delta = 20\log_{10}(\delta)\ \text{dB} \quad (13)$$

where $\delta = \min[\delta_p, \delta_s]$. This problem is commonly solved with the Parks-McClellan algorithm [5]–[6]. This method is based on the Reméz exchange routine, the solution of which can be tested for optimality using the alternation theorem as described in [5]. Minimax filters present equiripple magnitude responses, as depicted in Figure 2 (dash-dotted curve).

3.2. Weighted Least-Squares Approach

The weighted least-squares (WLS) approach minimizes

$$\| E(\omega) \|_2^2 = \int_0^\pi |E(\omega)|^2 d\omega = \int_0^\pi W^2(\omega)|H(\omega) - \hat{H}(\omega)|^2 d\omega \quad (14)$$

With the discrete frequencies, (14) is approximated by

$$\| E(\omega) \|_2^2 \approx \mathbf{e}^T\mathbf{e} \quad (15)$$

the minimization of which is achieved with

$$\mathbf{a}^* = \left(\mathbf{U}^T\mathbf{W}^2\mathbf{U}\right)^{-1}\mathbf{U}^T\mathbf{W}^2\mathbf{h} \quad (16)$$

Referring to Figure 1, the WLS approach maximizes the passband-to-stopband ratio (PSR) of energies

$$PSR = 10\log_{10}\left(\frac{E_p}{E_s}\right)\ \text{dB} \quad (17)$$

where E_p and E_s are respectively defined as

$$E_p = 2\int_0^{\omega_p} |\hat{H}(\omega)|^2 d\omega, \quad E_s = 2\int_{\omega_s}^{\pi} |\hat{H}(\omega)|^2 d\omega \quad (18)$$

A typical lowpass nonrecursive digital filter designed with the WLS method is depicted in Figure 2 (solid curve), where the large ripples near the band edges are easily identified.

3.3. Lawson Algorithm

In 1961, Lawson derived a scheme that performs Chebyshev approximation as a limit of a special sequence of weighted least-p (L_p) approximations with p fixed. As applied to the nonrecursive digital-filter design problem, the L_2 Lawson

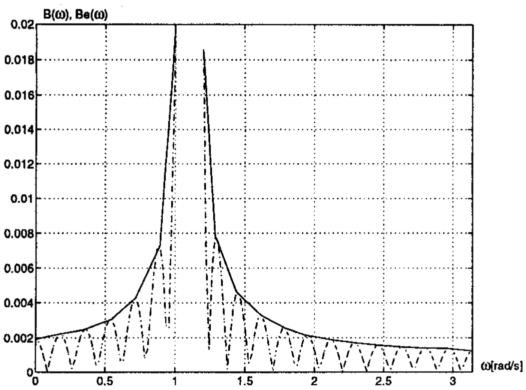

Figure 3. Absolute error function $B(\omega)$ (dash-dot) and corresponding envelope $B_e(\omega)$ (solid).

algorithm is implemented by a series of WLS approximations using a time-varying weight matrix $\mathbf{W}_k$, the elements of which are calculated by [2]

$$W_{k+1}^2(\omega) = W_k^2(\omega)B_k(\omega); \text{ with } B_k(\omega) = |E_k(\omega)| \quad (19)$$

Convergence of the Lawson algorithm is slow and some attempts to accelerate it are described in [2].

3.4. Lim-Lee-Chen-Yang Algorithm

An efficiently accelerated version of the Lawson algorithm was presented in [3]. The hereby referred to as the Lim-Lee-Chen-Yang (LLCY) algorithm is characterized by the weight matrix $\mathbf{W}_k$ recurrently updated by

$$W_{k+1}^2(\omega) = W_k^2(\omega)[Be_k(\omega)]^\theta \quad (20)$$

where θ is a scalar and $Be_k(\omega)$ is the envelope function of $B_k(\omega)$ formed by a set of piecewise linear segments that start and end at consecutive extremals of $B_k(\omega)$. Band edges are considered extremal frequencies, although edges from different bands should not be connected. In that manner, labeling the extremal frequencies as ω_J^*, for $J = 1, 2, \ldots$, the envelope function is formed as [3]

$$Be_k(\omega) = \frac{(\omega - \omega_J^*)B_k(\omega_{J+1}^*) + (\omega_{J+1}^* - \omega)B_k(\omega_J^*)}{(\omega_{J+1}^* - \omega_J^*)} \quad (21)$$

for all $\omega_J^* \leq \omega \leq \omega_{J+1}^*$. Figure 3 depicts typical cases of the absolute value of the error function (dash-dotted curve), used by the Lawson algorithm to update its weighting function, and its corresponding envelope (solid curve), used by the LLCY algorithm to update its weighting function.

4. A NEW APPROACH

Comparing the adjustments used by the Lawson and LLCY algorithms, described in (19)-(21) and seen in Figure 3, with the piecewise-constant weight function used by the WLS method, one can devise a very simple approach for designing nonrecursive digital filters that compromise both minimax and WLS constraints. The new approach consists of a modification on the weight-function updating procedure in such way that it becomes constant after a particular extremal of the stopband of $B_k(\omega)$, i.e.,

$$W_{k+1}^2(\omega) = W_k^2(\omega)[\beta_k(\omega)]^\theta \quad (22)$$

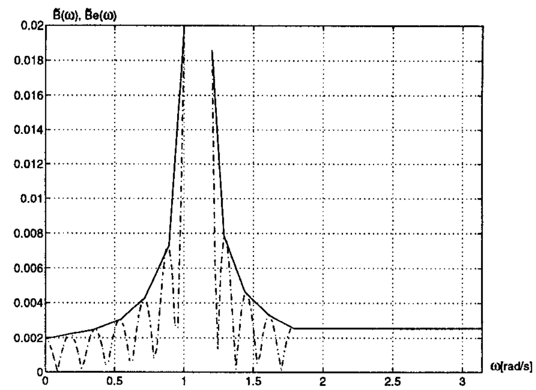

Figure 4. Modified functions for the new approach: Lawson (dash-dot) and LLCY (solid) versions.

where, for the Lawson and LLCY algorithms, $\beta_k(\omega)$ is respectively defined as

$$\beta_k(\omega) \equiv \tilde{B}_k(\omega) = \begin{cases} B_k(\omega), & \text{for } 0 \leq \omega \leq \omega_J^* \\ B_k(\omega_J^*), & \text{for } \omega_J^* < \omega \leq \pi \end{cases} \quad (23a)$$

$$\beta_k(\omega) \equiv \tilde{B}e_k(\omega) = \begin{cases} Be_k(\omega), & \text{for } 0 \leq \omega \leq \omega_J^* \\ Be_k(\omega_J^*), & \text{for } \omega_J^* < \omega \leq \pi \end{cases} \quad (23b)$$

where ω_J^* is the J-th extreme value of the stopband of $B(\omega) = |E(\omega)|$. The passband values of $B(\omega)$ and $Be(\omega)$ are left unchanged in equations (23a) and (23b) to preserve the equiripple property of the minimax method. An example of the new approach being applied to the functions seen in Figure 3 is depicted in Figure 4, where ω_J^* was chosen as the fifth extremal in the filter's stopband. The parameter J is the single design parameter for the proposed scheme. Choosing $J = 1$, turns the new scheme into an equiripple-passband WLS design. On the other hand, choosing J as large as possible, i.e., making $\omega_J^* = \pi$, turns the proposed scheme into the Lawson or LLCY algorithms.

The computational complexity of WLS-based algorithms is greatly reduced when one considers the Toeplitz-plus-Hankel internal structure of the matrix $(\mathbf{U}^T\mathbf{W}^2\mathbf{U})$ in (16), as mentioned in [7], and uses an efficient grid scheme to minimize the number of frequency values, as described in [8]. These simplifications make the computational complexity of WLS-based algorithms comparable to the one for the minimax approach. The WLS-based methods, however, do have the additional advantage of being easily implemented by simple computer routines.

5. NUMERICAL SIMULATIONS

Example 1: To illustrate the utilization of the proposed approach, a lowpass filter satisfying [1] $N = 95$, $DB_p = 1$ dB, $w_p = 2\pi 0.0625$ rad/s, and $w_s = 2\pi 0.0804$ rad/s was designed for all possible values of $1 \leq J \leq 42$. The resulting plot for DB_s and PSR, defined in (8b) and (17), respectively, is seen in Figure 5. From this figure, one can easily verify the poor results obtained with the minimax ($J = 42$) and WLS-like methods ($J = 1$), when considering both figures of merit simultaneously. The transfer function of the particular case when $J = 10$ is seen in Figure 6, from where one can notice the partially WLS-like and partially equiripple (up to its tenth extremal) stopband and the equiripple passband. These characteristics are typical

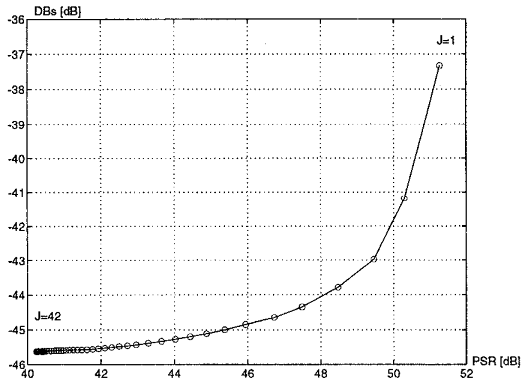

Figure 5. $DBs \times PSR$ **as functions of** J.

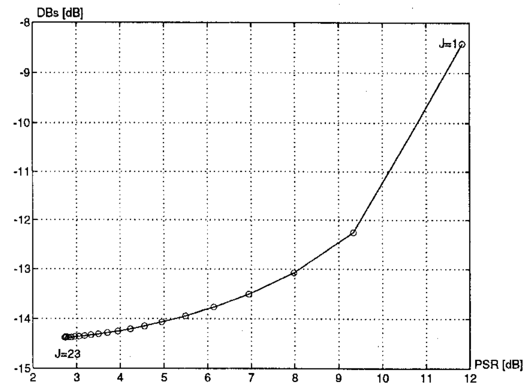

Figure 7. $DBs \times PSR$ **as functions of** J.

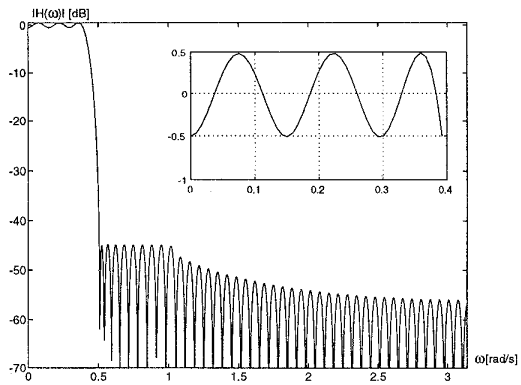

Figure 6. Lowpass frequency response for $J = 10$.

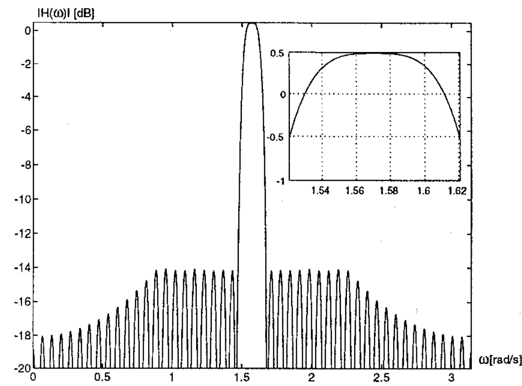

Figure 8. Bandpass frequency response for $J = 10$.

to the filters designed with the new approach. Both the Lawson and the LLCY variations were used yielding very similar results. The LLCY version, however, outperformed the modified-Lawson approach with respect to convergence speed, requiring fewer iterations to converge.

Example 2: In this example, a bandpass filter was designed with the new approach for all distinct values of $1 \leq J \leq 23$. The filter specifications were $DBp = 1$ dB, $w_{s1} = (\pi/2 - 0.1)$ rad/s, $w_{p1} = (\pi/2 - 0.05)$ rad/s, $w_{p2} = (\pi/2 + 0.05)$ rad/s, $w_{s2} = (\pi/2 + 0.1)$ rad/s, and $N = 95$. The plot of $DBs \times PSR$ for this design is shown in Figure 7, from which one can once more verify the poor performances achieved by the minimax ($J = 23$) and the WLS-like ($J = 1$) algorithms. The transfer function when $J = 10$ is seen in Figure 8.

6. CONCLUSION

A simple method for designing nonrecursive digital filters was presented. The method is based on a modification of the Lawson and Lim-Lee-Chen-Yang algorithms, forcing the weight function to become constant inside a frequency interval. The method's easy implementation along with the resulting combination of the minimax and WLS qualities indicate that it represents a very efficient form of compromising the stopband's peak and total energy constraints.

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