PEAK-CONSTRAINED LEAST-SQUARES DESIGN OF IIR DIGITAL FILTERS

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Paper Category: 1. Digital Filters, Wavelets, and Filter Banks. *

ABSTRACT: An approach for approximating infinite-duration impulse response (IIR) digital filters is proposed. The method is able to compromise maximum stopband attenuation and minimum stopband energy requirements. The approach is based on the weighted-least-squares (WLS) method with a weight function which is made constant within a given frequency interval at each iteration. In that sense, IIR digital filters with partially WLS-like and partially equiripple stopbands are efficiently designed.

1. INTRODUCTION

The Chebyshev and weighted-least-squares (WLS) methods are two well-known approaches for approximating digital filters. The Chebyshev scheme minimizes the maximum absolute value of a weighted error function between the prototype's transfer function and a given ideal solution. For that reason, the Chebyshev scheme is also referred to as the minimax approach. The WLS approach, which minimizes the mean-squared-value of the same weighted error function as the minimax approach, is characterized by a very simple implementation. Its basic problem, however, is the resulting Gibbs oscillations which correspond to large error near discontinuities of the desired response.

Some practical applications that use narrow-band filters, like frequency division multiplexing for communications, do require both the maximum stopband attenuation and the total stopband energy to be considered simultaneously. For these cases, Adams has shown [1] that both the minimax and the WLS approaches are unsuitable as they completely disregard one of these two measurements in their design procedure. For that matter, we propose a new approach for designing peak-constrained IIR digital filters with low stopband energy. For all practical purposes, the present work extends the results in [2] to the IIR case. In that sense, in Section 2, the general problem of designing IIR digital filters is presented. In Section 3, some classical optimization methods for solving that approximation problem are described. In Section 4, a new method is given resulting in an excellent compromise of all good properties of the minimax and WLS methods. Section 5 includes computer experiments demonstrating the good results achieved with the proposed method.

*This work was supported by CNPq - Ministry of Education, Brazil.

2. PROBLEM FORMULATION

Consider an IIR filter of order N described by the transfer function

$$\tilde{H}(z) = \frac{\tilde{B}(z)}{\tilde{A}(z)} = \frac{\sum_{j=0}^{N} \tilde{b}_j z^{-j}}{1 + \sum_{i=1}^{N} \tilde{a}_i z^{-i}} \tag{1}$$

The frequency response of such filter is

$$\tilde{H}(e^{j\omega}) = e^{j\hat{\theta}(\omega)} \hat{H}(\omega) \tag{2}$$

where $\hat{\theta}(\omega)$ and $\hat{H}(\omega)$ are the phase and magnitude responses of $\tilde{H}(e^{j\omega})$, respectively, defined as

$$\hat{\theta}(\omega) = \tan^{-1}\left\{\frac{\mathrm{Im}[\tilde{H}(e^{j\omega})]}{\mathrm{Re}[\tilde{H}(e^{j\omega})]}\right\} \tag{3a}$$

$$\hat{H}(\omega) = \left|\tilde{H}(e^{j\omega})\right| \tag{3b}$$

If the desired frequency response is given by

$$\tilde{H}_d(e^{j\omega}) = e^{j\hat{\theta}_d(\omega)} \hat{H}_d(\omega) \tag{4}$$

where $\hat{\theta}_d(\omega)$ and $\hat{H}_d(\omega)$ are defined similarly to $\hat{\theta}(\omega)$ and $\hat{H}(\omega)$ in (3), and $W(\omega)$ is a strictly positive weighting function, consider the weighted error function $\tilde{E}(e^{j\omega})$ defined in the frequency domain as

$$\tilde{E}(e^{j\omega}) = W(\omega)\left[\tilde{H}_d(e^{j\omega}) - \tilde{H}(e^{j\omega})\right] \tag{5}$$

The approximation problem for IIR digital filters resumes to the minimization of some objective function of $\tilde{E}(e^{j\omega})$ in such way that $|\tilde{E}(e^{j\omega})| \leq \delta$, and then

$$\left|\tilde{H}_d(e^{j\omega}) - \tilde{H}(e^{j\omega})\right| \leq \frac{\delta}{W(\omega)} \tag{6}$$

The error in (5), however, leads to a nonlinear optimization problem, and hence it is convenient to define an auxiliary error function as

$$\begin{aligned}\tilde{E}_B(e^{j\omega}) &= W(\omega)\left[\tilde{H}_d(e^{j\omega}) - \tilde{H}(e^{j\omega})\right] \\ &= W(\omega)\left[\tilde{H}_d(e^{j\omega})\tilde{A}(e^{j\omega}) - \tilde{B}(e^{j\omega})\right]\end{aligned} \tag{7}$$

Evaluating this auxiliary error function on a dense frequency grid with $0 \leq \omega_i \leq \pi$, for $i = 1, \dots, M \times N$, a good discrete approximation of $\tilde{E}_B(e^{j\omega})$ can be obtained. Points associated to the transition band are disregarded, and the remaining frequencies should be linearly redistributed in the passband and stopband to include their corresponding edges. Thus, the following vector equation holds

$$\tilde{\mathbf{e}} = \mathbf{W}\left(\tilde{\mathbf{h}} - \tilde{\mathbf{U}}\mathbf{a}\right) \tag{8}$$

where

$$\tilde{\mathbf{e}} = \left[\tilde{E}_B(e^{j\omega_1}) \;\dots\; \tilde{E}_B(e^{j\omega_{\tilde{C}}})\right]^T$$

$$\mathbf{W} = \mathrm{diag}\left[W(\omega_1) \;\dots\; W(\omega_{\tilde{C}})\right]$$

$$\tilde{\mathbf{h}} = \left[\tilde{H}(e^{j\omega_1}) \;\dots\; \tilde{H}(e^{j\omega_{\tilde{C}}})\right]^T$$

$$\tilde{\mathbf{U}} = \begin{bmatrix} 1 & e^{-j\omega_1} & \dots & e^{-jN\omega_1} & -e^{j\omega_1}\tilde{H}_d(e^{j\omega_1}) & \dots & -e^{jN\omega_1}\tilde{H}_d(e^{j\omega_1}) \\ 1 & e^{-j\omega_2} & \dots & e^{-jN\omega_2} & -e^{j\omega_2}\tilde{H}_d(e^{j\omega_2}) & \dots & -e^{jN\omega_2}\tilde{H}_d(e^{j\omega_2}) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-j\omega_{\tilde{C}}} & \dots & e^{-jN\omega_{\tilde{C}}} & -e^{j\omega_{\tilde{C}}}\tilde{H}_d(e^{j\omega_{\tilde{C}}}) & \dots & -e^{jN\omega_{\tilde{C}}}\tilde{H}_d(e^{j\omega_{\tilde{C}}}) \end{bmatrix}$$

$$\mathbf{a} = \left[\tilde{b}_0 \;\dots\; \tilde{b}_N \;\tilde{a}_1 \;\dots\; \tilde{a}_N\right]^T \tag{9}$$

with $\tilde{C} < M \times N$, as the original frequencies in the transition band were discarded.

An ideal lowpass filter is represented in Fig. 1, where δ_p is the passband maximum ripple, δ_s is the stopband minimum attenuation, and ω_p and ω_s are the passband and stopband edges, respectively. Based on these values, define

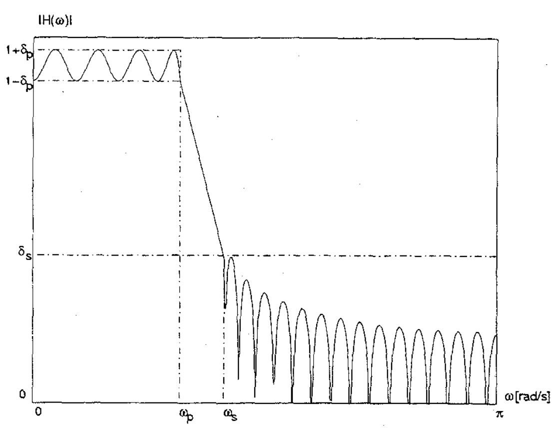

Figure 1. Typical lowpass filter specifications.

$$DB_p = 20\log_{10}\left(\frac{1+\delta_p}{1-\delta_p}\right) \text{ dB} \tag{10a}$$

$$DB_s = 20\log_{10}(\delta_s) \text{ dB} \tag{10b}$$

The design of a lowpass digital filter as specified in Fig. 1, using either the minimax method or the WLS approach, is

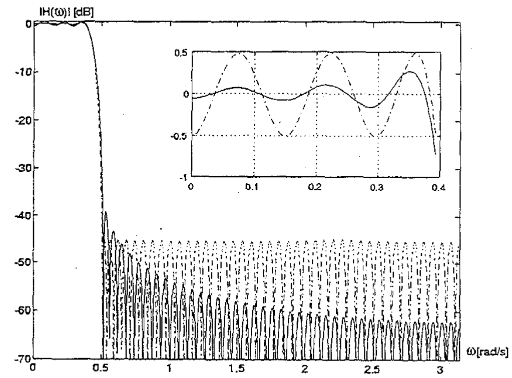

Figure 2. Lowpass frequency response of minimax- (dash-dotted) and WLS-based (solid) filters.

achieved making the ideal response and weight functions respectively equal to [3]

$$H(\omega) = \begin{cases} 1, & 0 \leq \omega \leq \omega_p \\ 0, & \omega_s \leq \omega \leq \pi \end{cases} \tag{11}$$

$$W(\omega) = \begin{cases} 1, & 0 \leq \omega \leq \omega_p \\ \delta_p/\delta_s, & \omega_s \leq \omega \leq \pi \end{cases} \tag{12}$$

3. OPTIMIZATION APPROACHES

3.1 Chebyshev Method

Chebyshev design consists on the minimization over the set of filter coefficients of the maximum value of $|\tilde{E}(e^{j\omega})|$, i.e.,

$$\|\tilde{E}(e^{j\omega})\|_\infty = \min_{\mathbf{a}} \max_{0\leq\omega\leq\pi} W(\omega)|\tilde{H}_d(e^{j\omega}) - \tilde{H}(e^{j\omega})| \tag{13}$$

Referring to Fig. 1, the minimax method optimizes

$$DB_\delta = 20\log_{10}(\delta) \text{ dB} \tag{14}$$

where $\delta = \min[\delta_p, \delta_s]$. A characteristic of minimax filters is their equiripple magnitude responses [4] as depicted in Figure 2 (dash-dotted curve).

3.2 Weighted-Least-Squares Method

The weighted least-squares (WLS) approach based on the auxiliary error defined in (6) minimizes

$$\begin{aligned}\|\tilde{E}_B(e^{j\omega})\|_2^2 &= \int_0^\pi W^2(\omega)|\tilde{H}_d(e^{j\omega}) - \tilde{H}(e^{j\omega})|^2 d\omega \\ &= \int_0^\pi W^2(\omega)|\tilde{H}_d(e^{j\omega})\tilde{A}(e^{j\omega}) - \tilde{B}(e^{j\omega})|^2 d\omega \end{aligned} \tag{15}$$

With the discrete frequencies, (15) is approximated by

$$\|\tilde{E}(e^{j\omega})\|_2^2 \approx \tilde{\mathbf{e}}^H\tilde{\mathbf{e}} \tag{16}$$

where ^{H} denotes the conjugate-transpose operator. The minimization of such function is achieved with

$$\mathbf{a}^* = \left[\mathrm{Re}(\tilde{\mathbf{U}}^H)\mathbf{W}\mathrm{Re}(\tilde{\mathbf{U}}) + \mathrm{Im}(\tilde{\mathbf{U}}^H)\mathbf{W}\mathrm{Im}(\tilde{\mathbf{U}})\right]^{-1} \left[\mathrm{Re}(\tilde{\mathbf{U}}^H)\mathbf{W}\mathrm{Re}(\tilde{\mathbf{h}}) + \mathrm{Im}(\tilde{\mathbf{U}}^H)\mathbf{W}\mathrm{Im}(\tilde{\mathbf{h}})\right] \quad (17)$$

Referring to Fig. 1, the WLS approach effectively maximizes the passband-to-stopband ratio (PSR) of energies

$$PSR = 10\log_{10}\left(\frac{\int_0^{\omega_p} \hat{H}^2(\omega)\mathrm{d}\omega}{\int_{\omega_s}^{\pi} \hat{H}^2(\omega)\mathrm{d}\omega}\right) \text{ dB} \quad (18)$$

A typical lowpass digital filter designed with the WLS method is depicted in Figure 2 (solid curve), where the large ripples near the band edges are easily identified.

3.3 Lawson Method

In 1961, Lawson derived a scheme that performs Chebyshev approximation as a limit of a special sequence of weighted least-p (L_p) approximations with p fixed. For instance, the L_2 Lawson algorithm is implemented by a series of WLS approximations using a time-varying weight matrix $\mathbf{W}_k$, the elements of which are calculated by [5]

$$W_{k+1}^2(\omega) = W_k^2(\omega)B_k(\omega) \quad (19)$$

with $B_k(\omega) = |\hat{E}_k(\omega)|$, where $\hat{E}_k(\omega)$ is an auxiliary error function defined as

$$\hat{E}_k(\omega) = W_k(\omega)\left[|\tilde{H}_d(e^{j\omega})| - |\tilde{H}(e^{j\omega})|\right] \quad (20)$$

3.4 Lim-Lee-Chen-Yang Method

An efficiently accelerated version of the Lawson algorithm was presented in [6]. The hereby referred to as the Lim-Lee-Chen-Yang (LLCY) algorithm is characterized by the weight matrix $\mathbf{W}_k$ recurrently updated by

$$W_{k+1}^2(\omega) = W_k^2(\omega)Be_k(\omega) \quad (21)$$

where $Be_k(\omega)$ is the envelope function of $B_k(\omega)$ formed by a set of piecewise linear segments that start and end at consecutive extremals of $B_k(\omega)$. Band edges are considered extremal frequencies, although edges from different bands should not be connected. In that manner, labeling the extremal frequencies as ω_J^*, for $J = 1, 2, \ldots$, the envelope function is formed as [6]

$$Be_k(\omega) = \frac{(\omega-\omega_J^*)B_k(\omega_{J+1}^*) + (\omega_{J+1}^* - \omega)B_k(\omega_J^*)}{(\omega_{J+1}^* - \omega_J^*)} \quad (22)$$

for all $\omega_J^* \leq \omega \leq \omega_{J+1}^*$.

Figure 3 depicts typical cases of the absolute value of the error function (dash-dotted curve), used by the Lawson algorithm to update its weighting function, and its corresponding envelope (solid curve), used by the LLCY algorithm to update its weighting function.

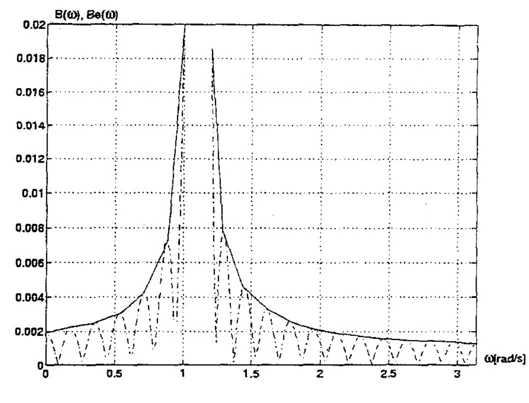

Figure 3. Absolute error function $B(\omega)$ (dash-dot) and corresponding envelope $B_e(\omega)$ (solid).

4. A NEW APPROACH

Comparing the adjustments used by the Lawson and LLCY algorithms, described in (19)–(22), with the piecewise-constant weight function used by the WLS method, one can devise a very simple approach for designing IIR digital filters that compromise both minimax and WLS constraints. The new approach consists of a modification on the weight-function update in such way that it becomes constant after a particular extremal ω_J^* of the stopband of $B_k(\omega)$, i.e.,

$$W_{k+1}^2(\omega) = W_k^2(\omega)\beta_k(\omega) \quad (23)$$

where, for the Lawson and LLCY algorithms, $\beta_k(\omega)$ is respectively defined as

$$\beta_k(\omega) \equiv \tilde{B}_k(\omega) = \begin{cases} B_k(\omega), & 0 \leq \omega \leq \omega_J^* \\ B_k(\omega_J^*), & \omega_J^* < \omega \leq \pi \end{cases} \quad (24a)$$

$$\beta_k(\omega) \equiv \tilde{B}e_k(\omega) = \begin{cases} Be_k(\omega), & 0 \leq \omega \leq \omega_J^* \\ Be_k(\omega_J^*), & \omega_J^* < \omega \leq \pi \end{cases} \quad (24b)$$

The passband values of $B_k(\omega)$ and $Be_k(\omega)$ are left unchanged in (24a) and (24b) to preserve the equiripple property of the minimax method. An example of the new approach is depicted in Fig. 4, where ω_J^* was chosen as the fifth extremal in the filter's stopband. The parameter J is the single design parameter for the proposed scheme. Choosing $J = 1$, turns the new scheme into an equiripple-passband WLS design. On the other hand, choosing J as large as possible, i.e., making $\omega_J^* = \pi$, turns the proposed scheme into the Lawson or LLCY algorithms.

5. COMPUTER SIMULATIONS

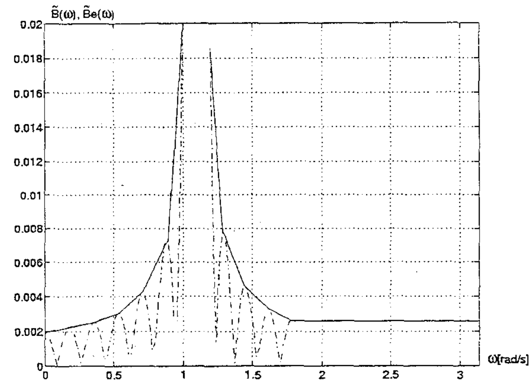

Figure 4. Modified weight functions for the new approach: Lawson (dash-dot) and LLCY (solid) versions.

Example 1: The proposed method is used to design an IIR filter specified as

$$\tilde{H}(e^{j\omega}) = \begin{cases} e^{-j\tau_s\omega}, & 0 \leq \omega \leq 0.4 \\ 0, & 0.6 \leq \omega \leq \pi \end{cases} \tag{25}$$

with $\tau_s = 12$, $N = 6$ and $DBp = 1.00$ dB. The case when $J = 3$, which is equivalent to the equiripple solution, resulted in $DBs = -53.6$ dB and a $PSR = 49.1$ dB. For $J = 2$, a good compromise between the figures of merit DBs and PSR could be achieved as their resulting values were -50.9 dB and 51.8 dB, respectively.

The resulting magnitude and group-delay responses for these two cases are shown in Fig. 5 and Fig. 6, with $J = 3$ (solid) and $J = 2$ (dash-dotted), respectively. Fig. 5 shows for these two cases a clear trade-off between the DBs and PSR figures of merit. In fig 6, notice how close the delay is in both cases to the specified value $\tau_s = 12$ within the filter's passband.

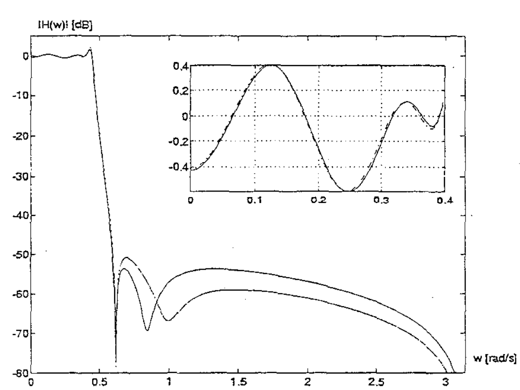

Figure 5. Ex. 1 - Magnitude responses (passband in detail) when $J = 3$ (solid) and $J = 2$ (dotted).

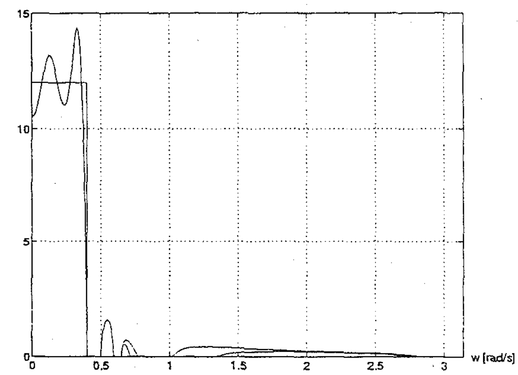

Figure 6. Ex. 1 - Group-delay responses (passband in detail) when $J = 3$ (solid) and $J = 2$ (dotted).

6. CONCLUSION

A simple method for designing IIR digital filters was presented. The method is based on a modification of the so-called Lawson and Lim-Lee-Chen-Yang algorithms, forcing the weight function to become constant inside a frequency interval. The method's easy implementation along with the resulting combination of the minimax and WLS qualities yield an excellent compromise of the stopband's peak and energy requirements.

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