Designing Peak-Constrained Arbitrary-Phase FIR Digital Filters with Low Passband-to-Stopband Energy Ratio

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Abstract: An approach for approximating nonrecursive digital filters with arbitrary phase is proposed. The method compromises maximum stopband attenuation and minimum stopband energy. The approach is based on a time-varying weight function which is made constant, at each iteration, inside a given frequency interval. In that sense, a partially WLS-like and partially equiripple stopband is achieved.

1. Introduction

The design of nonrecursive digital filters is dominated in the literature by the Chebyshev and the weighted least-squares (WLS) approaches. The Chebyshev scheme minimizes the maximum absolute value of a weighted error function between the prototype's transfer function and a given ideal solution. For that reason, the Chebyshev scheme is also referred to as the minimax approach. The WLS approach, which minimizes the mean-squared-value of the same weighted error function as the minimax approach, is characterized by a very simple implementation. Its basic problem, however, is the resulting Gibbs oscillations which correspond to large error near discontinuities of the desired response.

Some practical applications that use narrow-band filters, like frequency division multiplexing for communications, do require both the maximum stopband attenuation and the total stopband energy to be considered simultaneously. For these cases, Adams has shown [1] that both the minimax and the WLS approaches are unsuitable as they disregard one of these two measurements in their design procedure. For that matter, we propose a new approach for designing peak-constrained nonrecursive digital filters with low passband-to-stopband energy ratio. The present work extends the results in [2] to the arbitrary-phase case. For that matter, in Section 2, the general problem of designing nonrecursive digital filters is presented. In Section 3, the classical optimization methods for solving that approximation problem are described. In Section 4, a new method is given resulting in an excellent compromise of all good properties of the minimax and WLS methods. Section 5 includes computer experiments demonstrating the good results achieved with the proposed method.

2. Problem Formulation

Description of the approximation problem of an arbitrary-phase nonrecursive digital filter is a bit distinct from the linear-phase case, thus requiring a notation system slightly different from the one used in [2]. In fact, consider a nonrecursive filter of length N described by the transfer function

$$\tilde{H}(z) = \sum_{n=0}^{N-1} h(nT) z^{-n} \tag{1}$$

and assume that $\omega_s = 2\pi$, such that $T = 1$. The frequency response of such filter is then given by

$$\tilde{H}(e^{j\omega}) = \sum_{n=0}^{N-1} h(n) e^{-j\omega n} = e^{-j\hat{\theta}(\omega)} \hat{H}(\omega) \tag{2}$$

where $\hat{\theta}(\omega)$ and $\hat{H}(\omega)$ are the phase and magnitude responses of $\tilde{H}(e^{j\omega})$ respectively defined as

$$\hat{\theta}(\omega) = \tan^{-1}\left\{\frac{\text{Im}[\tilde{H}(e^{j\omega})]}{\text{Re}[\tilde{H}(e^{j\omega})]}\right\} \tag{3a}$$

$$\hat{H}(\omega) = \left|\tilde{H}(e^{j\omega})\right| \tag{3b}$$

If the desired frequency response is given by

$$\tilde{H}_d(e^{j\omega}) = e^{-j\hat{\theta}_d(\omega)} \hat{H}_d(\omega) \tag{4}$$

where $\hat{\theta}_d(\omega)$ and $\hat{H}_d(\omega)$ are defined similarly to $\hat{\theta}(\omega)$ and $\hat{H}(\omega)$ in (3), and $W(\omega)$ is a strictly positive weighting function, consider the weighted error function $\tilde{E}(e^{j\omega})$ defined in the frequency domain as

$$\tilde{E}(e^{j\omega}) = W(\omega)\left[\tilde{H}_d(e^{j\omega}) - \tilde{H}(e^{j\omega})\right] \tag{5}$$

The approximation problem for nonrecursive digital filters resumes to the minimization of some objective function of $\tilde{E}(e^{j\omega})$ in such way that $|\tilde{E}(e^{j\omega})| \leq \delta$, and then

$$\left|\tilde{H}_d(e^{j\omega}) - \tilde{H}(e^{j\omega})\right| \leq \frac{\delta}{W(\omega)} \tag{6}$$

Evaluating the weighted error function on a dense frequency grid with $0 \leq \omega_i \leq \pi$, for $i = 1, \ldots, M \times N$, a good discrete approximation of $\tilde{E}(e^{j\omega})$ can be obtained. Points associated to the transition band are disregarded, and the remaining frequencies should be linearly redistributed in the passband and stopband to include their corresponding edges. Thus, the following vector equation holds

$$\tilde{\mathbf{e}} = \mathbf{W}\left(\tilde{\mathbf{h}} - \tilde{\mathbf{U}}\mathbf{a}\right) \tag{7}$$

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where

$$\tilde{\mathbf{e}} = \left[\tilde{E}(e^{j\omega_1}) \ \dots \ \tilde{E}(e^{j\omega_{\bar{C}}})\right]^T \quad (8a)$$

$$\mathbf{W} = \text{diag}\left[W(\omega_1) \ \dots \ W(\omega_{\bar{C}})\right] \quad (8b)$$

$$\tilde{\mathbf{h}} = \left[\tilde{H}(e^{j\omega_1}) \ \dots \ \tilde{H}(e^{j\omega_{\bar{C}}})\right]^T \quad (8c)$$

$$\tilde{\mathbf{U}} = \begin{bmatrix} 1 & e^{-j\omega_1} & \dots & e^{-Nj\omega_1} \\ 1 & e^{-j\omega_2} & \dots & e^{-Nj\omega_2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-j\omega_{\bar{C}}} & \dots & e^{-Nj\omega_{\bar{C}}} \end{bmatrix} \quad (8d)$$

$$\mathbf{a} = \left[h(0) \ \dots \ h(N-1)\right]^T \quad (8e)$$

with $\bar{C} < M \times N$, as the original frequencies in the transition band were discarded.

An ideal lowpass filter is represented in Fig. 1, where δ_p is the passband maximum ripple, δ_s is the stopband minimum attenuation, and ω_p and ω_s are the passband and stopband edges, respectively. Based on these values,

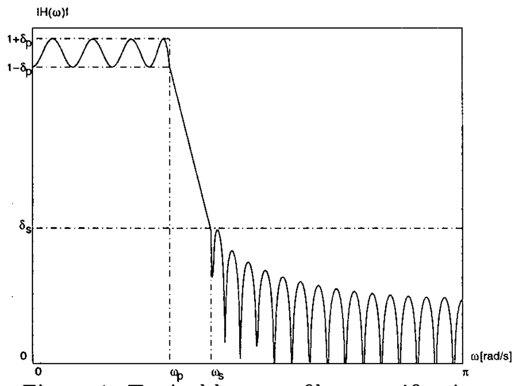

Figure 1: Typical lowpass filter specifications.

define

$$DB_p = 20\log_{10}\left(\frac{1+\delta_p}{1-\delta_p}\right) \text{ dB} \quad (9a)$$

$$DB_s = 20\log_{10}(\delta_s) \text{ dB} \quad (9b)$$

The design of a lowpass digital filter as specified in Fig. 1, using either the minimax method or the WLS approach, is achieved making the ideal response and weight functions respectively equal to [5]

$$H(\omega) = \begin{cases} 1, & 0 \le \omega \le \omega_p \\ 0, & \omega_s \le \omega \le \pi \end{cases} \quad (10)$$

$$W(\omega) = \begin{cases} 1, & 0 \le \omega \le \omega_p \\ \delta_p/\delta_s, & \omega_s \le \omega \le \pi \end{cases} \quad (11)$$

3. Optimization Approaches

3.1 Chebyshev Method

Chebyshev design consists on the minimization over the set of filter coefficients of the maximum value of $|\tilde{E}(e^{j\omega})|$, i.e.,

$$\|\tilde{E}(e^{j\omega})\|_\infty = \min_{\mathbf{a}} \max_{0\le\omega\le\pi} W(\omega)|\tilde{H}_d(e^{j\omega}) - \tilde{H}(e^{j\omega})| \quad (12)$$

With the discrete set of frequencies, using equation (8), the minimax function becomes

$$\|\tilde{E}(e^{j\omega})\|_\infty \approx \min_{\mathbf{a}} \max_{0\le\omega_i\le\pi} \mathbf{W}|\tilde{\mathbf{h}} - \tilde{\mathbf{U}}\mathbf{a}| \quad (13)$$

Referring to Fig. 1, the minimax method optimizes

$$DB_\delta = 20\log_{10}(\delta) \text{ dB} \quad (14)$$

where $\delta = \min[\delta_p, \delta_s]$. A characteristic of minimax filters is their equiripple magnitude responses [6].

3.2 Weighted-Least-Squares Method

The weighted least-squares (WLS) approach minimizes

$$\|\tilde{E}(e^{j\omega})\|_2^2 = \int_0^\pi W^2(\omega)|\tilde{H}_d(e^{j\omega}) - \tilde{H}(e^{j\omega})|^2 d\omega \quad (15)$$

With the discrete frequencies, (15) is approximated by

$$\|\tilde{E}(e^{j\omega})\|_2^2 \approx \tilde{\mathbf{e}}^H \tilde{\mathbf{e}} \quad (16)$$

where ^{H} denotes the conjugate-transpose operator. The minimization of such function is achieved with

$$\mathbf{a}^* = \left[\text{Re}(\tilde{\mathbf{U}}^H)\ \mathbf{W}\ \text{Re}(\tilde{\mathbf{U}}) + \text{Im}(\tilde{\mathbf{U}}^H)\ \mathbf{W}\ \text{Im}(\tilde{\mathbf{U}})\right]^{-1} \left[\text{Re}(\tilde{\mathbf{U}}^H)\ \mathbf{W}\ \text{Re}(\tilde{\mathbf{h}}) + \text{Im}(\tilde{\mathbf{U}}^H)\ \mathbf{W}\ \text{Im}(\tilde{\mathbf{h}})\right] \quad (17)$$

Referring to Fig. 1, the WLS approach maximizes the passband-to-stopband ratio (PSR) of energies

$$PSR = 10\log_{10}\left(\frac{\int_0^{\omega_p} \hat{H}^2(\omega)d\omega}{\int_{\omega_s}^{\pi} \hat{H}^2(\omega)d\omega}\right) \text{ dB} \quad (18)$$

3.3 Lawson Method

In 1961, Lawson derived a scheme that performs Chebyshev approximation as a limit of a special sequence of weighted least-p (L_p) approximations with p fixed. As applied to the nonrecursive digital-filter design problem, the L_2 Lawson algorithm is implemented by a series of WLS approximations using a time-varying weight matrix $\mathbf{W}_k$, the elements of which are calculated by [3]

$$W_{k+1}^2(\omega) = W_k^2(\omega)B_k(\omega) \quad (19)$$

with $B_k(\omega) = |\hat{E}_k(\omega)|$, where $\hat{E}_k(\omega)$ is an auxiliary error function defined as

$$\hat{E}_k(\omega) = W_k(\omega)\left[|\tilde{H}_d(e^{j\omega})| - |\tilde{H}(e^{j\omega})|\right] \quad (20)$$

The method based on the modified error $\hat{E}_k(\omega)$ consistently outperformed the scheme based on the original error given in (5), with respect to the algorithm's stability and the characteristics of the final solution, as illustrated in Ex. 3 below.

3.4 Lim-Lee-Chen-Yang Method

An efficiently accelerated version of the Lawson algorithm was presented in [4]. The hereby referred to as the Lim-Lee-Chen-Yang (LLCY) algorithm is characterized by the weight matrix $\mathbf{W}_k$ recurrently updated by

$$W_{k+1}^2(\omega) = W_k^2(\omega)Be_k(\omega) \quad (21)$$

where $Be_k(\omega)$ is the envelope function of $B_k(\omega)$ formed by a set of piecewise linear segments that start and end at consecutive extremals of $B_k(\omega)$. Band edges are considered extremal frequencies, although edges from different bands should not be connected. In that manner, labeling the extremal frequencies as ω_J^*, for $J = 1, 2, \dots$, the envelope function is formed as [4]

$$Be_k(\omega) = \frac{(\omega - \omega_J^*)B_k(\omega_{J+1}^*) + (\omega_{J+1}^* - \omega)B_k(\omega_J^*)}{(\omega_{J+1}^* - \omega_J^*)} \quad (22)$$

for all $\omega_J^* \leq \omega \leq \omega_{J+1}^*$.

4. A New Approach

Comparing the adjustments used by the Lawson and LLCY algorithms, described in (19)–(22), with the piecewise-constant weight function used by the WLS method, one can devise a very simple approach for designing nonrecursive digital filters that compromise both minimax and WLS constraints. The new approach consists of a modification on the weight-function update in such way that it becomes constant after a particular extremal ω_J^* of the stopband of $B_k(\omega)$, i.e.,

$$W_{k+1}^2(\omega) = W_k^2(\omega)\beta_k(\omega) \tag{23}$$

where, for the Lawson and LLCY algorithms, $\beta_k(\omega)$ is respectively defined as

$$\beta_k(\omega) \equiv \tilde{B}_k(\omega) = \begin{cases} B_k(\omega), & 0 \leq \omega \leq \omega_J^* \\ B_k(\omega_J^*), & \omega_J^* < \omega \leq \pi \end{cases} \tag{24a}$$

$$\beta_k(\omega) \equiv \tilde{B}e_k(\omega) = \begin{cases} Be_k(\omega), & 0 \leq \omega \leq \omega_J^* \\ Be_k(\omega_J^*), & \omega_J^* < \omega \leq \pi \end{cases} \tag{24b}$$

The passband values of $B_k(\omega)$ and $Be_k(\omega)$ are left unchanged in (24a) and (24b) to preserve the equiripple property of the minimax method. An example of the new approach is depicted in Fig. 2, where ω_J^* was chosen as the fifth extremal in the filter's stopband. The parameter

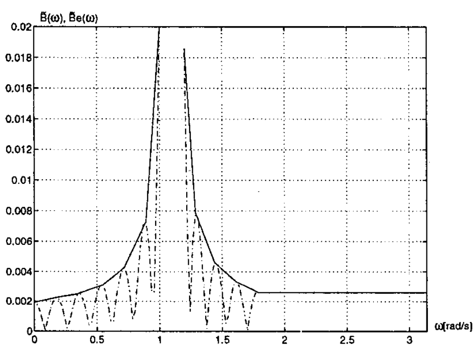

Figure 2: Examples of weight functions defined by the new approach.

J is the single design parameter for the proposed scheme. Choosing $J = 1$, turns the new scheme into an equiripple-passband WLS design. On the other hand, choosing J as large as possible, i.e., making $\omega_J^* = \pi$, turns the proposed scheme into the Lawson or LLCY algorithms.

5. Numerical Simulations

Example 1: The proposed method is used to design a nonrecursive filter specified as [7]

$$\tilde{H}(e^{j\omega}) = \begin{cases} e^{-j\tau_s\omega}, & 0 \leq \omega \leq 0.12\pi \\ 0, & 0.24\pi \leq \omega \leq \pi \end{cases} \tag{25}$$

with $\tau_s = 12$, $N = 31$ and $\delta_p/\delta_s = 10$. The case when $J = 14$, which is equivalent to the equiripple solution, resulted in $\delta_p = 0.03538$ and $\delta_s = 0.003536$. These values are considerably better than the results mentioned in [7]. Using $J = 3$, we obtain $\delta_p = 0.04427$ and $\delta_s = 0.004423$, which are comparable to the results in [7], with an additional 3.2 dB for the PSR.

The resulting magnitude and group-delay responses are shown in Fig. 3 and Fig. 4, with $J = 3$ (solid) and $J = 14$ (dash-dotted), respectively. Notice how close the delay is in both cases to the specified value $\tau_s = 12$ within the filter's passband.

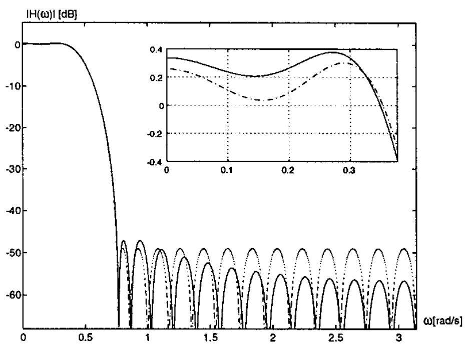

Figure 3: Ex. 1 - Magnitude response (passband in detail) when $J = 3$ (solid) and $J = 14$ (dash-dotted).

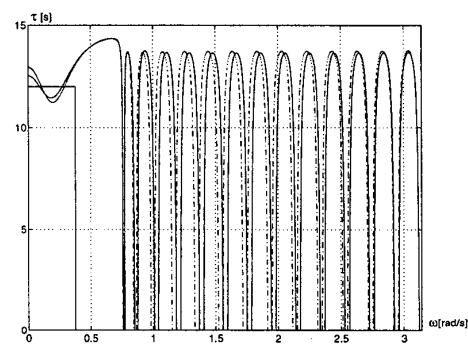

Figure 4: Ex. 1 - Group-delay response when $J = 3$ (solid) and $J = 14$ (dash-dotted).

Example 2: The approximation of arbitrary-phase nonrecursive filters require that both the desired magnitude and phase responses to be specified. If, however, only the magnitude response is given, a minimum-phase response can be derived using the Hilbert transform, as described in [8]. This approach was used here to design a lowpass filter defined by $\omega_p = 0.4$ rad/s, $\omega_s = 0.6$ rad/s, $N = 71$, $DBp = 1$ dB, and $DBs = -30$ dB. The desired (dash-dotted) and obtained (solid) magnitude and phase responses are shown in Fig. 5 and Fig. 6, respectively. Notice that the magnitude response satisfies the prescribed specifications and the obtained phase response coincides with the desired one within the filter's passband, for all practical purposes.

Example 3: To illustrate how the modified error defined in (20) results in a better scheme than the error defined in (5), the two arbitrary-phase filters described in Ex. 1 and Ex. 2 were approximated using these two error functions. Convergence of each scheme was monitored through the DBp measurement as depicted in Fig. 7 and Fig. 8, respectively. For the filter given in Ex. 1, when $N = 35$ the modified error clearly converged to a filter with smaller passband ripple, whereas for other values of

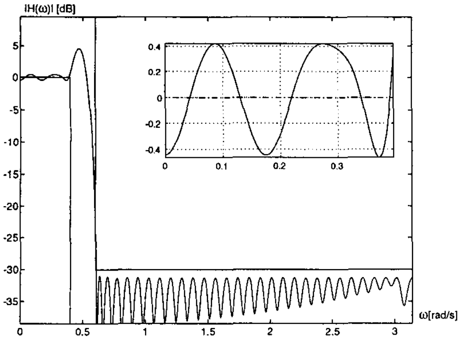

Figure 5: Ex. 2 - Magnitude response (passband in detail).

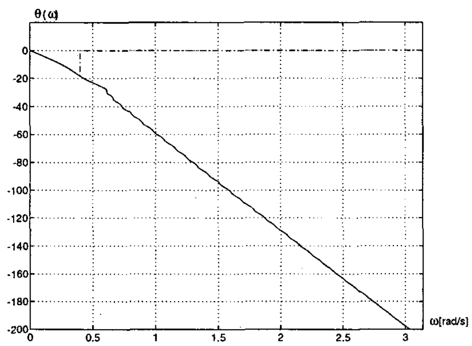

Figure 6: Ex. 2 - Desired (dash-dotted) and obtained (solid) phase responses.

N both schemes yielded similar results. For the filter in Ex. 2, however, the modified error consistently outperformed the original error, which presented a somewhat erratic convergence.

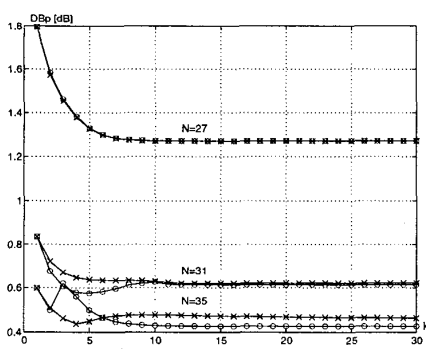

Figure 7: Ex. 3 - DBp convergence for the design in Ex. 1 with N = 27, 31, 35 using the modified ('o') and the original ('×') error functions.

6. Conclusion

A simple method for designing nonrecursive digital filters

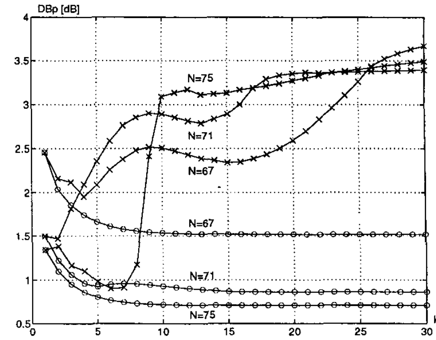

Figure 8: Ex. 3 - DBp convergence for the design in Ex. 2 with N = 67, 71, 75 using the modified ('o') and the original ('×') error functions.

with arbitrary phase was presented. The method is based on a modification of the Lawson and Lim-Lee-Chen-Yang algorithms, forcing the weight function to become constant inside a frequency interval. The method's easy implementation along with the resulting combination of the minimax and WLS qualities indicate that it represents a very efficient form of compromising the stopband's peak and energy constraints.

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