ON WLS-CHEBYSHEV IIR DIGITAL FILTERS

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ABSTRACT: A method for approximating infinite-duration impulse response (IIR) digital filters is given. The method is able to compromise maximum stopband attenuation and minimum stopband energy requirements. The proposed approach performs a series of weighted-least-squares (WLS) iterations, with a weight function which is made constant within a given frequency interval at each iteration. The highly nonlinear problem characteristic of IIR filter design is pseudo-linearized using the Steiglitz-McBride formulation. As a result, IIR digital filters with partially WLS-like and partially equiripple stopbands are efficiently designed.

1. INTRODUCTION

The Chebyshev and weighted-least-squares (WLS) methods are two well-known approaches for approximating digital filters. A characteristic of the Chebyshev scheme is the minimization of the maximum absolute value of a weighted error function between the resulting transfer function and a given ideal response. Solution of the Chebyshev problem is commonly performed with routines which are very computationally demanding. Meanwhile, the WLS approach, which minimizes the mean-squared-value of the same weighted error function as the Chebyshev method, is characterized by a very simple implementation. Its main disadvantage, however, is the resulting Gibbs oscillations which correspond to large error near discontinuities of the desired response.

Some practical applications that use narrow-band filters, like frequency division multiplexing for communications, do require both the maximum stopband attenuation and the total stopband energy to be considered simultaneously. For these cases, Adams has shown [1] that both the minimax and the WLS approaches are unsuitable, as their design solutions completely disregard one of these two measurements. For that matter, a new approach is proposed for designing peak-constrained IIR digital filters with low stopband energy. The present work extends the results in [2] and [3] to the IIR case using an alternative definition for the error function. In fact, here the design problem of IIR digital filters is described using the Steiglitz-McBride formulation [4], [5] that yields a pseudo-linearized optimization problem. The resulting method is then characterized by an extremely simple implementation based on a series of WLS designs with a different weight function at each iteration. These weighting functions are made constant for a given frequency interval resulting in a very good compromise between the characteristics of the WLS and Chebyshev solutions.

This paper is organized as follows: In Section 2, the general problem of designing IIR digital filters is presented. In Section 3, some classical optimization methods for solving that approximation problem are described. In Section 4, the new method is given yielding a good compromise between the stopband's minimum attenuation and total energy. Section 5 then includes filter designs illustrating the good results achieved with the proposed approach.

2. IIR PROBLEM FORMULATION

Consider an IIR filter of order N described by the transfer function

$$\tilde{H}(z) = \frac{\tilde{B}(z)}{\tilde{A}(z)} = \frac{\sum_{j=0}^{N} \tilde{b}_j z^{-j}}{1 + \sum_{i=1}^{N} \tilde{a}_i z^{-i}} \tag{1}$$

The frequency response of such filter is

$$\tilde{H}(e^{j\omega}) = e^{j\hat{\theta}(\omega)} \hat{H}(\omega) \tag{2}$$

where $\hat{\theta}(\omega)$ and $\hat{H}(\omega)$ are the phase and magnitude responses of $\tilde{H}(e^{j\omega})$, defined as

$$\hat{\theta}(\omega) = \tan^{-1} \left\{ \frac{\mathrm{Im}[\tilde{H}(e^{j\omega})]}{\mathrm{Re}[\tilde{H}(e^{j\omega})]} \right\} \tag{3a}$$

$$\hat{H}(\omega) = \left| \tilde{H}(e^{j\omega}) \right| \tag{3b}$$

respectively. We can then write the desired frequency response as

$$\tilde{H}_d(e^{j\omega}) = e^{j\hat{\theta}_d(\omega)} \hat{H}_d(\omega) \tag{4}$$

where $\hat{\theta}_d(\omega)$ and $\hat{H}_d(\omega)$ are defined similarly to $\hat{\theta}(\omega)$ and $\hat{H}(\omega)$ in (3). In addition, we define the error function $\tilde{E}(e^{j\omega})$ as

$$\tilde{E}(e^{j\omega}) = W(\omega) \left[\tilde{H}_d(e^{j\omega}) - \tilde{H}(e^{j\omega}) \right] \tag{5}$$

where $W(\omega)$ is a strictly positive weighting function. The approximation problem for IIR digital filters thus is reduced to the minimization of some objective function of $\tilde{E}(e^{j\omega})$ in such way that $|\tilde{E}(e^{j\omega})| \leq \delta$, and then

$$\left| \tilde{H}_d(e^{j\omega}) - \tilde{H}(e^{j\omega}) \right| \leq \frac{\delta}{W(\omega)} \tag{6}$$

The error function in (5), however, leads to a nonlinear optimization problem, and hence it is convenient to use an

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alternative error function. In [3] a definition corresponding to the equation error (EE) scheme was employed. However, it is well known in the adaptive filtering literature that the EE approach may lead to unstable solutions in problems of system identification when in the presence of measurement or modeling additive noise. Such behavior also occurred in [3] when the EE scheme was used in the filter design set up. For that matter, in here we choose the error at the k-th iteration defined as

$$\begin{aligned}\tilde{E}_{SM,k}(e^{j\omega}) &= W(\omega)\frac{\tilde{A}_k(e^{j\omega})}{\tilde{A}_{k-1}(e^{j\omega})}\left[\tilde{H}_d(e^{j\omega}) - \tilde{H}_k(e^{j\omega})\right] \\ &= \frac{W(\omega)}{\tilde{A}_{k-1}(e^{j\omega})}\left[\tilde{A}_k(e^{j\omega})\tilde{H}_d(e^{j\omega}) - \tilde{B}_k(e^{j\omega})\right] \quad (7)\end{aligned}$$

This error formulation corresponds to the Steiglitz-McBride (SM) approach, which was originally devised to the problems of system identification and continuous-time filter design [4], [5], but has also been applied to design digital filters based on the WLS method [6]. Such choice was based on the fact that the SM approach is less affected by the problem of unstable solution than the EE scheme. Other additional modifications are currently being considered to further improve the convergence properties of the overall design procedure.

Evaluating the alternative error function on a dense frequency grid with $0 \leq \omega_i \leq \pi$, for $i = 1, \ldots, M \times N$, a good discrete approximation of $\tilde{E}_{SM,k}(e^{j\omega})$ can be obtained. Thus, the following equation holds

$$\tilde{\mathbf{e}} = \mathbf{W}\left(\tilde{\mathbf{h}} - \tilde{\mathbf{U}}\mathbf{a}\right) \quad (8)$$

where

$$\tilde{\mathbf{e}} = \left[\tilde{E}_{SM}(e^{j\omega_1}) \; \ldots \; \tilde{E}_{SM}(e^{j\omega_{\bar{C}}})\right]^T$$

$$\mathbf{W} = \mathrm{diag}\left[W_k(\omega_1) \; \ldots \; W_k(\omega_{\bar{C}})\right]$$

$$\tilde{\mathbf{h}} = \left[\tilde{H}_k(e^{j\omega_1}) \; \ldots \; \tilde{H}_k(e^{j\omega_{\bar{C}}})\right]^T$$

$$\tilde{\mathbf{U}} = \begin{bmatrix} 1 & e^{-j\omega_1} & \ldots & e^{-jN\omega_1} & -e^{j\omega_1}\tilde{H}_d(e^{j\omega_1}) & \ldots & -e^{jN\omega_1}\tilde{H}_d(e^{j\omega_1}) \\ 1 & e^{-j\omega_2} & \ldots & e^{-jN\omega_2} & -e^{j\omega_2}\tilde{H}_d(e^{j\omega_2}) & \ldots & -e^{jN\omega_2}\tilde{H}_d(e^{j\omega_2}) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-j\omega_{\bar{C}}} & \ldots & e^{-jN\omega_{\bar{C}}} & -e^{j\omega_{\bar{C}}}\tilde{H}_d(e^{j\omega_{\bar{C}}}) & \ldots & -e^{jN\omega_{\bar{C}}}\tilde{H}_d(e^{j\omega_{\bar{C}}}) \end{bmatrix}$$

$$\mathbf{a} = \left[\tilde{b}_0 \; \ldots \; \tilde{b}_N \; \tilde{a}_1 \; \ldots \; \tilde{a}_N\right]^T \quad (9)$$

where $\bar{C} < M \times N$, as the original frequencies in the transition band were discarded, and $W_k(\omega)$ is defined as

$$W_k(\omega) = \frac{W(\omega)}{|\tilde{A}_{k-1}(e^{j\omega})|} \quad (10)$$

An ideal lowpass filter is represented in Fig. 1, where δ_p is the passband maximum ripple, δ_s is the stopband minimum attenuation, and ω_p and ω_s are the passband and stopband edges, respectively. Based on these values, define

$$DB_p = 20\log_{10}\left(\frac{1+\delta_p}{1-\delta_p}\right) \text{ dB} \quad (11a)$$

$$DB_s = 20\log_{10}(\delta_s) \text{ dB} \quad (11b)$$

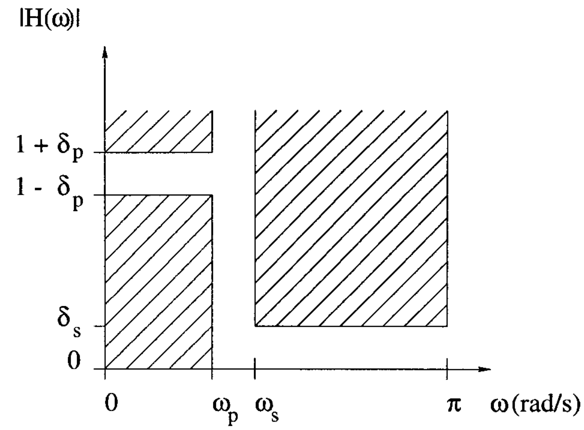

Figure 1. Typical lowpass filter specifications.

The design of a lowpass digital filter as specified in Fig. 1, using either the minimax method or the WLS approach, is achieved making the ideal response and weight functions respectively equal to [7]

$$\hat{H}_d(\omega) = \begin{cases} 1, & 0 \leq \omega \leq \omega_p \\ 0, & \omega_s \leq \omega \leq \pi \end{cases} \quad (12)$$

$$W(\omega) = \begin{cases} 1, & 0 \leq \omega \leq \omega_p \\ \delta_p/\delta_s, & \omega_s \leq \omega \leq \pi \end{cases} \quad (13)$$

3. CLASSIC APPROACHES

3.1 Chebyshev Method

Chebyshev design consists on the minimization over the set of filter coefficients of the maximum value of $|\tilde{E}(e^{j\omega})|$, i.e.,

$$\|\tilde{E}(e^{j\omega})\|_\infty = \min_{\mathbf{a}} \max_{0\leq\omega\leq\pi} W(\omega)|\tilde{H}_d(e^{j\omega}) - \tilde{H}(e^{j\omega})| \quad (14)$$

Referring to Fig. 1, the minimax method minimizes

$$DB_\delta = 20\log_{10}(\delta) \text{ dB} \quad (15)$$

where $\delta = \max[\delta_p, \delta_s]$. A characteristic of minimax filters is their equiripple magnitude responses [8] as depicted in Fig. 2 (dash-dotted curve).

3.2 Weighted-Least-Squares Method

The weighted least-squares (WLS) approach based on the auxiliary error defined in (6) minimizes at the k-th iteration

$$\|\tilde{E}_{SM,k}(e^{j\omega})\|_2^2 = \int_0^\pi W^2(\omega)\frac{|\tilde{A}_k(e^{j\omega})|^2}{|\tilde{A}_{k-1}(e^{j\omega})|^2}|\tilde{H}_d(e^{j\omega}) - \tilde{H}(e^{j\omega})|^2 d\omega \quad (16)$$

With the discrete frequencies, (16) is approximated by

$$\|\tilde{E}_{SM}(e^{j\omega})\|_2^2 \approx \tilde{\mathbf{e}}^H\tilde{\mathbf{e}} \quad (17)$$

where ^{H} denotes the conjugate-transpose operator. The minimization of such function is achieved by

$$\begin{aligned}\mathbf{a}^*_{SM} = &\left[\mathrm{Re}(\tilde{\mathbf{U}}^H)\mathbf{W}\mathrm{Re}(\tilde{\mathbf{U}}) + \mathrm{Im}(\tilde{\mathbf{U}}^H)\mathbf{W}\mathrm{Im}(\tilde{\mathbf{U}})\right]^{-1} \cdot \\ &\left[\mathrm{Re}(\tilde{\mathbf{U}}^H)\mathbf{W}\mathrm{Re}(\tilde{\mathbf{h}}) + \mathrm{Im}(\tilde{\mathbf{U}}^H)\mathbf{W}\mathrm{Im}(\tilde{\mathbf{h}})\right] \quad (18)\end{aligned}$$

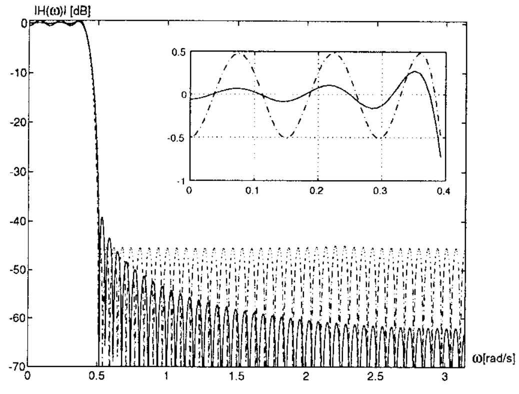

Figure 2. Lowpass frequency response of minimax- (dash-dotted) and WLS-based (solid) filters.

Referring to Fig. 1, the WLS approach ideally maximizes the passband-to-stopband ratio (PSR) of energies

$$PSR = 10\log_{10}\left(\frac{\int_0^{\omega_p} \hat{H}^2(\omega)\mathrm{d}\omega}{\int_{\omega_s}^{\pi} \hat{H}^2(\omega)\mathrm{d}\omega}\right) \text{ dB} \tag{19}$$

A typical lowpass digital filter designed with the WLS method is depicted in Fig. 2 (solid curve), where the large ripples near the band edges are easily identified.

3.3 Lim-Lee-Chen-Yang Method

In 1961, Lawson derived a scheme that performs Chebyshev approximation as a limit of a special sequence of weighted least-p (L_p) approximations with p fixed. For instance, the L_2 Lawson algorithm is implemented by a series of WLS approximations using a varying weight matrix $\mathbf{W}_k$, the elements of which are calculated by [9]

$$W_{k+1}^2(\omega) = W_k^2(\omega)B_k(\omega) \tag{20}$$

with $B_k(\omega) = |\hat{E}_{SM,k}(\omega)|$, where $\hat{E}_{SM,k}(\omega)$ is the alternative error function at the k-th iteration.

The Lim-Lee-Chen-Yang (LLCY)algorithm [10] is an efficiently accelerated version of the Lawson algorithm. That approach is characterized by the weight matrix $\mathbf{W}_k$ recurrently updated as

$$W_{k+1}^2(\omega) = W_k^2(\omega)Be_k(\omega) \tag{21}$$

where $Be_k(\omega)$ is the envelope function of $B_k(\omega)$ formed by a set of piecewise linear segments that start and end at consecutive extremals of $B_k(\omega)$. Band edges are considered extremal frequencies, although edges from different bands should not be connected.

Fig. 3 depicts typical cases of the absolute value of the error function (dash-dotted curve), used by the Lawson algorithm to update its weighting function, and its corresponding envelope (solid curve), used by the LLCY algorithm to update its weighting function.

4. THE NEW APPROACH

Comparing the adjustments used by the Lawson and LLCY algorithms, described in (20)–(21), with the

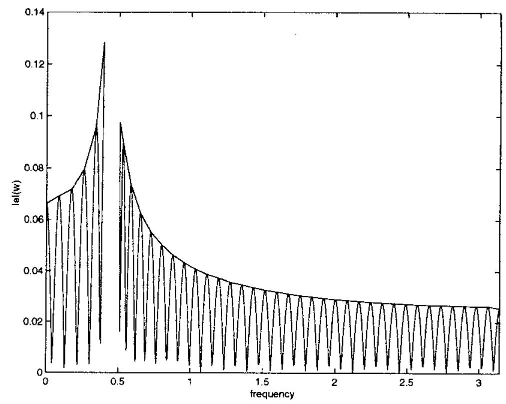

Figure 3. Absolute error function $B(\omega)$ (dash-dot) and corresponding envelope $B_e(\omega)$ (solid).

piecewise-constant weight function used by the WLS method, one can devise a very simple approach for designing IIR digital filters that compromises both minimax and WLS constraints. The new approach consists of a modification on the weight-function update in such way that it becomes constant after a particular extremal ω_J^* of the stopband of $B_k(\omega)$, i.e.,

$$W_{k+1}^2(\omega) = W_k^2(\omega)\beta_k(\omega) \tag{22}$$

where, for the Lawson and LLCY algorithms, $\beta_k(\omega)$ is respectively defined as

$$\beta_k(\omega) \equiv \tilde{B}_k(\omega) = \begin{cases} B_k(\omega), & 0 \leq \omega \leq \omega_J^* \\ B_k(\omega_J^*), & \omega_J^* < \omega \leq \pi \end{cases} \tag{23a}$$

$$\beta_k(\omega) \equiv \tilde{B}e_k(\omega) = \begin{cases} Be_k(\omega), & 0 \leq \omega \leq \omega_J^* \\ Be_k(\omega_J^*), & \omega_J^* < \omega \leq \pi \end{cases} \tag{23b}$$

An example of the new approach is depicted in Fig. 4, where ω_J^* was chosen as the fifteenth extremal in the filter's stopband. One should note the slight distinction between Fig. 3

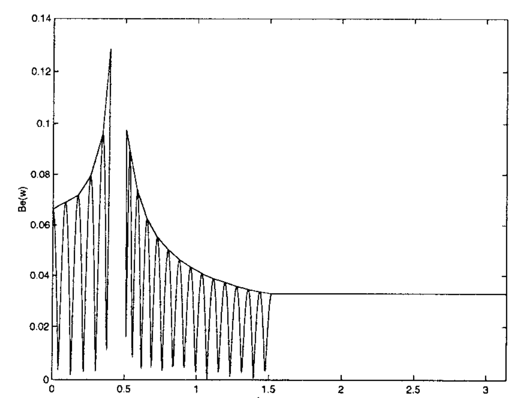

Figure 4. Modified weight functions for the new approach: Lawson (dash-dot) and LLCY (solid) versions.

and Fig. 4 for $\omega \geq 1.5$ rad/s.

The parameter J is the single design parameter for the proposed scheme. Choosing $J = 1$, turns the new scheme into an equiripple-passband WLS design. On the other hand, choosing J as large as possible, i.e., making $\omega_J^* = \pi$, turns the proposed scheme into the Lawson or LLCY algorithms. Our experience has indicated that choosing $J \approx P/5$, where P is the total number of ripples in the stopband, yields a very good compromise between the levels of stopband energy and minimum attenuation.

5. COMPUTER SIMULATION

The proposed method is used to design an IIR filter specified as

$$\tilde{H}(e^{j\omega}) = \begin{cases} e^{-j\tau_s\omega}, & 0 \leq \omega \leq 1.4 \\ 0, & 1.5 \leq \omega \leq \pi \end{cases} \quad (24)$$

with $\tau_s = 12$, $N = 12$ and $DBp = 0.1$ dB.

The resulting magnitude and group-delay responses (within the passband) are shown in Fig. 5 and Fig. 6, respectively, for different values of J. Fig. 5 shows a clear trade-off between the DBs and PSR figures of merit. More specifically, for $J = 4$ (Chebyshev design), the DBs was -35.38 dB and $PSR = 36.97$ dB, and for $J = 3$, we had $DBs = -33.00$ dB and $PSR = 39.03$ dB. For the other values of J, the specifications were not satisfied. In fig 6, notice how close the delay is in all cases to the specified value $\tau_s = 12$.

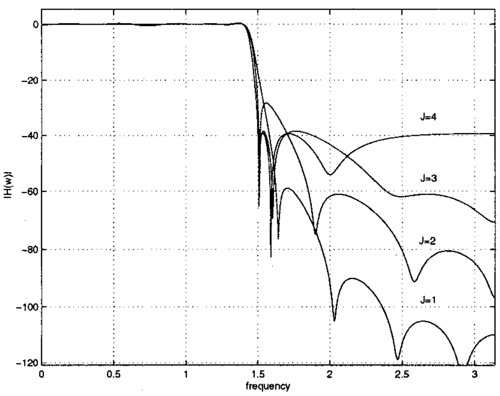

Figure 5. Magnitude responses for different values of J.

As reported in [10] and [11], which present alternative methods to approximate IIR digital filters, the resulting filter may not be stable and the algorithm may not converge. In our simulations, these facts were observed when the filter order was higher than necessary to satisfy the prescribed specifications. Therefore, if some numerical instability is observed during the convergence process, the filter order should be reduced and the process started all over again. Some additional strategies are currently under consideration to further improve the convergence properties of the overall design procedure.

6. CONCLUSION

A simple method for designing IIR digital filters was presented. The method is based on a modification of the so-called Lawson and Lim-Lee-Chen-Yang algorithms, forcing the weight function to become constant in a specific frequency interval. The Steiglitz-McBride approach was used to pseudo-linearize the overall optimization problem. In

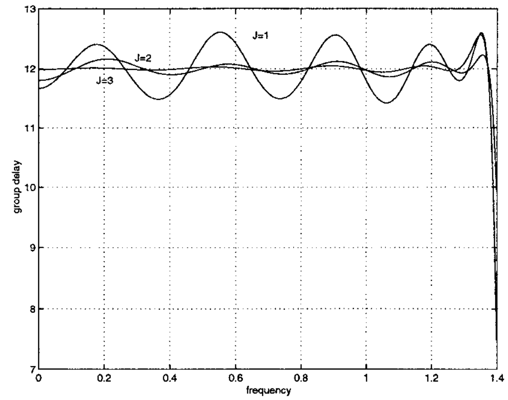

Figure 6. Passband group-delay responses for different values of J.

that manner, the modified Chebyshev problem was solved through a series of WLS iterations with different weight functions for each iteration. The result is an IIR digital filter whose stopband presents good characteristics with respect to the levels of attenuation and energy, simultaneously.

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