BEAMSPACE COVARIANCE-BASED DOA ESTIMATION

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ABSTRACT

This work presents a new version, with reduced computational complexity, of the covariance-based direction-of-arrival (CB-DoA) algorithm. The new algorithm incorporates the concept of beamspace projection before performing the DoA estimation. Such modification reduces the dimensions of the matrices employed by the elementspace CB-DoA, simplifying the resulting computations while preserving the detectability of the original algorithm. The Beamspace CB-DoA algorithm is compared to the traditional algorithm Beamspace ESPRIT, as well as to elementspace CB-DoA.

1. INTRODUCTION

In mobile communication systems, the concept of cellular division has emerged as a new paradigm. Cellular-based systems present some advantages in comparison to the previous single cell systems, such as serving more users, using less power during transmission, and requiring a comparatively narrower bandwidth [1]. In order to effectively reduce the bandwidth, the logical channels are reused in non-neighbouring cells. This is illustrated in Fig. 1, where cells with the same pattern use the same range of logical channels. In this figure, for the mobile represented in the central cell, the antenna transmitting in a similar non-neighbouring cell is a potential source of the so-called co-channel interference (CCI).

Spatial filtering has recently emerged as a powerful alternative to mitigate CCI. When the location of the desired source is known, information coming from all other positions can be reduced greatly.

Consider a communications system with multiple transmitting sources and multiple receiving antennas. Suppose the transmition medium to be isotropic, the sources to be located in the far field of the receiving array, and the sources and the receiving array to be co-planar. With these assumptions, localizing the sources is equivalent to measuring the direction-of-arrival (DoA) of the receiving waves, as represented in Fig. 2, where the angle θ is the parameter to be estimated.

The first DoA estimation algorithms were based on the maximum likelihood (ML) concept, which is very computationally demanding [2]. Later, some sub-optimal alternatives to ML were developed, such as ESPRIT (estimation of parameters via rotational invariance techniques) [3], which present less computational requirements. In order to decrease the number of operations, ESPRIT imposes an aditional constraint over the geometry of the receiving antennas: The receiving array is divided into pairs of antennas (doublets) with a constant displacement between them.

Based on matrix pencil methods for harmonic retrieval [4], the elementspace CB-DoA algorithm [5] is a simpler alternative to ESPRIT, imposing the same constraints to the receiving array geometry. A further simplifying technique, commonly referred to as the beamspace approach [6] [7], may be regarded as sectorizing the range of angles of arrival, specially when DFT beamspace is used. Such sectorization allows parallel computation of the sub-bands.

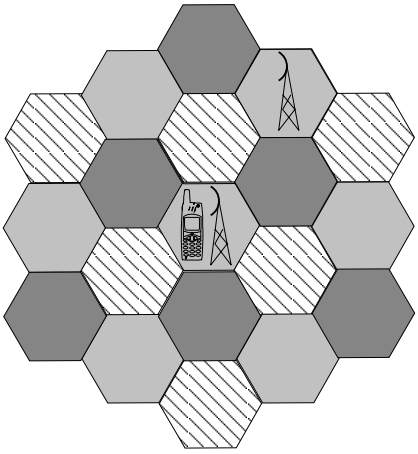

Fig. 1. Frequency reuse for a cell-based system.

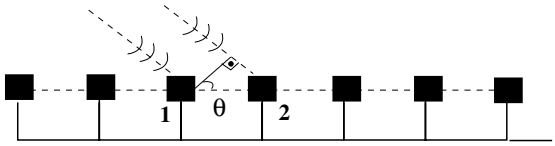

Fig. 2. Angle estimated by DoA algorithms.

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The main objective of this article is to incorporate this beamspace concept to the CB-DoA algorithm to further decrease its computational complexity. For that matter, this paper is structured as following described: In Sections 2 and 3, the beamspace versions of the ESPRIT and CB-DoA algorithms are described, respectively. The computational complexities of these two algoritms are compared in Section 4 and Section 5. Section 6 presents simulation results achieved by the proposed technique, comparing its performance with the Beamspace ESPRIT algorithm and the theoretical Cramer-Rao Lower Bound (CRLB).

2. BEAMSPACE ESPRIT

Consider a MIMO (multiple-input multiple-output) environment with M transmitting narrowband sources and N receiving antennas, with $N > M$, as represented in Fig. 3. Moreover, assume that each sub-channel suffers interference from an additive white Gaussian noise (AWGN) with variance σ_N^2. The antennas in the receiving array are constrained to be uniformly spaced. At time t, let $s_m(t)$ represent the signal trans-

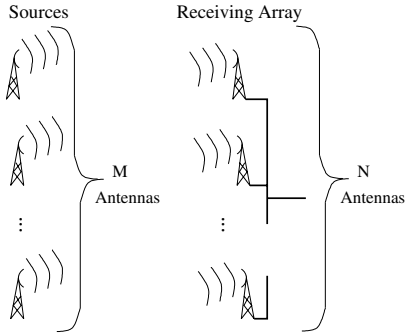

Fig. 3. MIMO Environment.

mitted by the m^{th} antenna, with $0 \leq m \leq (M-1)$, and let $x_i(t)$ be the signal impinging in the i^{th} receiving antenna, with $0 \leq i \leq (N-1)$. Considering that the incoming signal from the m^{th} source reaches the i^{th} antenna with an angle denoted by θ_m, the gain provided by the antenna for such an angle is represented by $a_i(\theta_m)$.

If $n_{x,i}(t)$ represents the noise components received by the i^{th} antenna, the description of the received signals as functions of the transmitted signals is given by [3]

$$x_i(t) = \sum_{m=0}^{M-1} s_m(t) a_i(\theta_m) + n_{x,i}(t). \tag{1}$$

By defining the following auxiliary vectors and matrices in the discrete-time domain k

$$\mathbf{x}(k) = \begin{bmatrix} x_0(k) & x_1(k) & \dots & x_{N-1}(k) \end{bmatrix}^{\mathrm{T}}, \tag{2}$$

$$\mathbf{n}_x(k) = \begin{bmatrix} n_{x,0}(k) & n_{x,1}(k) & \dots & n_{x,N-1}(k) \end{bmatrix}^{\mathrm{T}}, \tag{3}$$

$$\mathbf{s}(k) = \begin{bmatrix} s_0(k) & s_1(k) & \dots & s_{M-1}(k) \end{bmatrix}^{\mathrm{T}}, \tag{4}$$

$$\mathbf{A} = \begin{bmatrix} a_0(\theta_0) & a_0(\theta_1) & \dots & a_0(\theta_{M-1}) \\ a_1(\theta_0) & a_1(\theta_1) & \dots & a_1(\theta_{M-1}) \\ \vdots & \vdots & \ddots & \vdots \\ a_{N-1}(\theta_0) & a_{N-1}(\theta_1) & \dots & a_{N-1}(\theta_{M-1}) \end{bmatrix}, \tag{5}$$

then the input-to-output relationships given in Equation (1) can be rewritten as

$$\mathbf{x}(k) = \mathbf{A}\mathbf{s}(k) + \mathbf{n}_x(k). \tag{6}$$

Consider selective matrices $\mathbf{J}_1$ and $\mathbf{J}_2$ defined by

$$\mathbf{J}_1 = \begin{bmatrix} \mathbf{I}_{\{K\times K\}} & \mathbf{0}_{\{K\times(N-K)\}} \end{bmatrix}, \tag{7}$$

$$\mathbf{J}_2 = \begin{bmatrix} \mathbf{0}_{\{K\times(N-K)\}} & \mathbf{I}_{\{K\times K\}} \end{bmatrix}, \tag{8}$$

such that $\mathbf{J}_1\mathbf{J}_1^H = \mathbf{J}_2\mathbf{J}_2^H = \mathbf{I}$, where $\mathbf{I}$ denotes an identity sub-matrix and $\mathbf{0}$ denotes a sub-matrix containing only zeros.

If δ is the constant displacement between adjacent receiving antennas, matrix $\mathbf{A}$ has a Vandermonde structure, giving rise to the expression

$$\mathbf{J}_1\mathbf{A} = \mathbf{J}_2\mathbf{A}\mathbf{\Phi}^H, \tag{9}$$

with

$$\mathbf{\Phi} = \mathrm{diag}\left[e^{\frac{j\omega\delta}{c}\sin(\theta_0)}, e^{\frac{j\omega\delta}{c}\sin(\theta_1)}, \dots, e^{\frac{j\omega\delta}{c}\sin(\theta_{M-1})}\right], \tag{10}$$

where ω is the frequency of the narrowband signal and c is the speed of light.

In the beamspace approach, the signal vector $\mathbf{x}(k)$ is processed by orthogonal transforms $\mathbf{T}_i$, with dimensions $N \times L$, where $L < N$. Then, considering that the AWGN is uncorrelated to the sources, a covariance model for the system is

$$\mathbf{R}_x = \mathbf{T}_i^H\mathbf{A}\mathbf{R}_s\mathbf{A}^H\mathbf{T}_i + \sigma_N^2\mathbf{I}, \tag{11}$$

where $\mathbf{R}_x$ and $\mathbf{R}_s$ denote the autocorrelation matrices of the received and transmitted signals, respectively.

For the Beamspace ESPRIT to work properly, $\mathbf{T}_i\mathbf{A}$ must retain the rotational invariance property of $\mathbf{A}$. In [6], the transform $\mathbf{T}_i$ can be applied to $\mathbf{x}(k)$ if it satisfies

$$\mathbf{J}_1\mathbf{T}_i = \mathbf{J}_2\mathbf{T}_i\mathbf{F}, \tag{12}$$

for a full-rank $L \times L$ matrix $\mathbf{F}$. Let $\mathbf{t}_i$, for $0 \leq i \leq (N-1)$, be the i^{th} column of $\mathbf{T}_i^H$. If a matrix $\mathbf{Q}$ exists, such that

$$\begin{cases} \mathbf{Q}\mathbf{F}^H\mathbf{t}_i = 0, & 0 \leq i < N-K, \\ \mathbf{Q}\mathbf{t}_i = 0, & K \leq i < N, \end{cases} \tag{13}$$

then one can write, using equations (9) and (12) and the basic properties of $\mathbf{J}_1$ and $\mathbf{J}_2$, that

$$\begin{aligned} \mathbf{Q}\mathbf{T}_i^H\mathbf{A} &= \mathbf{Q}\mathbf{T}_i^H\mathbf{J}_1^H\mathbf{J}_1\mathbf{A} \\ &= \mathbf{Q}\mathbf{T}_i^H\mathbf{J}_1^H\mathbf{J}_2\mathbf{A}\mathbf{\Phi}^H \\ &= \mathbf{Q}\mathbf{F}^H\mathbf{T}_i^H\mathbf{J}_2^H\mathbf{J}_2\mathbf{A}\mathbf{\Phi}^H \\ &= \mathbf{Q}\mathbf{F}^H\mathbf{T}_i^H\mathbf{A}\mathbf{\Phi}^H. \end{aligned} \tag{14}$$

Then, an eigendecomposition is performed on $\mathbf{R}_x$, and its eigenvectors belonging to the signal subspace are grouped in matrix $\mathbf{E}_s$. Therefore, a new invariance equation is reached from equation (14),

$$\mathbf{Q}\mathbf{T}_i^H\mathbf{E}_s = \mathbf{Q}\mathbf{F}^H\mathbf{T}_i^H\mathbf{E}_s\mathbf{\Phi}^H, \tag{15}$$

which may be solved in a total-least-squares (TLS) sense, i.e., defining

$$\begin{bmatrix}\mathbf{E}_1 & \mathbf{E}_2\end{bmatrix} = \begin{bmatrix}\mathbf{Q}\mathbf{T}_i^H\mathbf{E}_s & \mathbf{Q}\mathbf{F}^H\mathbf{T}_i^H\mathbf{E}_s\end{bmatrix}, \tag{16}$$

then compute the eigendecomposition of

$$\begin{bmatrix}\mathbf{E}_1^H \\ \mathbf{E}_2^H\end{bmatrix}\begin{bmatrix}\mathbf{E}_1 & \mathbf{E}_2\end{bmatrix} = \mathbf{E}\mathbf{\Lambda}\mathbf{E}^H. \tag{17}$$

Matrix $\mathbf{E}$ is partitioned into

$$\mathbf{E} = \begin{bmatrix}\mathbf{E}_{11} & \mathbf{E}_{12} \\ \mathbf{E}_{21} & \mathbf{E}_{22}\end{bmatrix}. \tag{18}$$

Matrix $\mathbf{\Phi}$, which represents the rotational invariance, is computed based on an EVD decomposition on $\mathbf{\Psi} = -\mathbf{E}_{12}\mathbf{E}_{22}^{-1}$.

3. BEAMSPACE CB-DOA

Consider a schematic modeling for the system represented in Fig. 4. Received data are projected in different subspaces by matrices $\mathbf{T}_i$ and later processed by CB-DoA algorithm. Consider two correlation matrices $\mathbf{R}_1$ and $\mathbf{R}_2$ such that

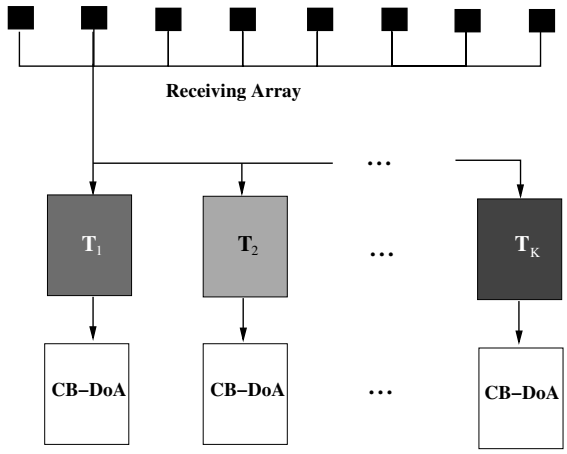

Fig. 4. General modeling for Beamspace CB-DoA.

$$\mathbf{R}_1 = \mathbf{Q}\mathbf{R}_x\mathbf{Q}^H = \mathbf{Q}[\mathbf{T}_i^H\mathbf{A}\mathbf{R}_s\mathbf{A}^H\mathbf{T}_i - \sigma_N^2\mathbf{I}]\mathbf{Q}^H, \tag{19}$$

$$\mathbf{R}_2 = \mathbf{Q}\mathbf{F}^H\mathbf{R}_x\mathbf{Q}^H, \tag{20}$$

where, in equation (19), the estimation of the noise correlation is extracted from the signal correlation. By performing an eigendecomposition on $\mathbf{R}_1$, one gets

$$\mathbf{R}_1 = \mathbf{U}\mathbf{\Sigma}^2\mathbf{U}. \tag{21}$$

Let $\mathbf{U}_s$ be the eigenvectors of $\mathbf{R}_1$ belonging to the signal subspace and $\mathbf{\Sigma}_s^2$ their corresponding eigenvalues. One may conclude that $\mathbf{Q}\mathbf{T}_i^H\mathbf{A}$ and $\mathbf{U}_s\mathbf{\Sigma}_s$ span the same subspace, that is,

$$\mathbf{Q}\mathbf{T}_i^H\mathbf{A} = \mathbf{U}_s\mathbf{\Sigma}_s\mathbf{V}, \tag{22}$$

for a full-rank matrix $\mathbf{V}$.

Defining the auxiliary matrix $\mathbf{F}_s$ as

$$\mathbf{F}_s = \mathbf{\Sigma}_s^{-1}\mathbf{U}_s^H, \tag{23}$$

and using equations (11) and (14), then an auxiliary matrix $\mathbf{R}_2$ is determined as

$$\begin{aligned}\mathbf{R}_2 &= \mathbf{F}_s\mathbf{Q}\mathbf{F}^H(\mathbf{R}_x - \sigma_N^2\mathbf{I})\mathbf{Q}^H\mathbf{F}_s^H \\ &= \mathbf{\Sigma}_s^{-1}\mathbf{U}_s^H\mathbf{Q}\mathbf{F}^H(\mathbf{R}_x - \sigma_N^2\mathbf{I})\mathbf{Q}^H\mathbf{U}_s\mathbf{\Sigma}_s \\ &= \mathbf{\Sigma}_s^{-1}\mathbf{U}_s^H\mathbf{Q}\mathbf{F}^H\mathbf{T}_i^H\mathbf{A}\mathbf{A}^H\mathbf{T}_i\mathbf{Q}^H\mathbf{U}_s\mathbf{\Sigma}_s \\ &= \mathbf{\Sigma}_s^{-1}\mathbf{U}_s^H\mathbf{Q}\mathbf{T}_i^H\mathbf{A}\mathbf{\Phi}\mathbf{A}^H\mathbf{T}_i\mathbf{Q}^H\mathbf{U}_s\mathbf{\Sigma}_s\end{aligned} \tag{24}$$

Hence, from equation (22), one may conclude that

$$\mathbf{R}_2 = \mathbf{V}\mathbf{\Phi}\mathbf{V}^H, \tag{25}$$

that is, $\mathbf{\Phi}$ may be found by an eigendecomposition (EVD) of $\mathbf{R}_2$.

4. COMPARISON TO BEAMSPACE ESPRIT

In order to allow the comparison between the Beamspace ESPRIT and Beamspace CB-DoA algorithms, the computational complexity of both methods is investigated in this section. For that purpose, Table 1 summarizes the basic operations for each algorithm. When referring to multiple rows or columns, Matlab notation was used. Recalling that M is the number of sources, N is the number of sensors, and L is the beamspace dimension, the number of operations for each algorithm can be determined. Beamspace algorithm is based on the use of different sectorization for estimating the DoA angles. Nevertheless, as presented in [6], the different beamspace projections may be performed in parallel, due to its inherent modularization. Therefore, the computational complexity analysis is performed on only 1 beamspace projection, for both algorithms.

From Table 1, one verifies that the Beamspace ESPRIT algorithm requires:

- 3 eigendecompositions (1 for a $2M \times 2M$ Hermitian matrix, 1 for an $M \times M$ Hermitian matrix, and 1 for an $L \times L$ Hermitian matrix);
- 1 full-matrix inversion of an $M \times M$ matrix;
- 10 matrix multiplications (2 for the product of an $L \times L$ and $L \times N$ matrices, 2 for the product of an $L \times L$ and $L \times N$ matrices, 1 for the product of a pair of $L \times L$ matrices, and 4 for the product of an $M \times L$ and an $L \times M$ matrices).

Table 1. Short descriptions of Beamspace ESPRIT and Beamspace CB-DoA algorithms.

Beams. ESPRIT	Beams. CB-DoA
$[\mathbf{U}_s, \hat{\sigma}_N^2] = \text{EVD}(\hat{\mathbf{R}}_x)$	$[\mathbf{U}_s, \hat{\sigma}_N^2] = \text{EVD}(\mathbf{Q}\hat{\mathbf{R}}_x\mathbf{Q}^H)$
$\mathbf{E}_x = \mathbf{Q}\mathbf{T}_i^H\mathbf{U}_s$	$\mathbf{F} = \mathbf{\Sigma}_s^{-1}\mathbf{U}_s^H$
$\mathbf{E}_y = \mathbf{Q}\mathbf{F}^H\mathbf{T}_i^H\mathbf{U}_s$	$\mathbf{R}_a = \hat{\mathbf{R}}_x - \hat{\sigma}^2\mathbf{I}$
$\mathbf{E}_a = \begin{bmatrix}\mathbf{E}_x^H \\ \mathbf{E}_y^H\end{bmatrix}\begin{bmatrix}\mathbf{E}_x & \mathbf{E}_y\end{bmatrix}$	$\mathbf{R}_1 = \mathbf{F}_s\mathbf{Q}\mathbf{F}^H\mathbf{R}_a\mathbf{Q}^H\mathbf{F}_s^H$
$[\mathbf{E}, \mathbf{\Lambda}] = \text{EVD}(\mathbf{E}_a)$	$[\mathbf{\Phi}^H] = \text{EVD}(\mathbf{R}_1)$
$\mathbf{E}_{12} = \mathbf{E}(0 : M-1, M : \text{end})$	
$\mathbf{E}_{22} = \mathbf{E}(M : \text{end}, M : \text{end})$	
$\mathbf{\Psi} = -\mathbf{E}_{12}\mathbf{E}_{22}^{-1}$	
$[\mathbf{\Phi}] = \text{EVD}(\mathbf{\Psi})$	

On the other hand, the new CB-DoA method requires:

- 2 eigendecompositions (1 for an $L \times L$ Hermitian matrix, and 1 for an $M \times M$ Hermitian matrix);
- 1 diagonal-matrix inversion of an $M \times M$ matrix;
- 8 matrix multiplications (2 for products of pair of $L \times L$ matrices, 1 for a product of an $M \times M$ and an $M \times L$ matrices, and 4 for products of an $M \times L$ and $L \times L$ matrices and 1 for a product of an $M \times L$ and an $L \times M$ matrices);
- 1 matrix subtraction of a pair of $N \times N$ matrices.

Although the computational cost of each method is highly dependent on implementation issues, it is straightforward to verify that the proposed algorithm presents smaller complexity than the Beamspace ESPRIT. In fact, the Beamspace CB-DoA requires fewer matrix multiplications, includes a simpler (diagonal) matrix inversion, and requires less eigendecompositions, which is a very computationally intensive operation. In Table 2, a detailed comparison is provided which shows the asymptotic complexity of the basic operations, according to [8].

Table 2. Number of operations required by Beamspace ESPRIT and Beamspace CB-DoA.

Operation	Compl. [8]	B. ESPRIT	B. CB-DoA
Herm. Eigendec.	$\mathcal{O}(n^2)$	3	2
Full Inversion	$\mathcal{O}(2n^3/3)$	1	–
Diag. Inversion	$\mathcal{O}(n)$	–	1
Multiplication	$\mathcal{O}(n^3)$	10	8
Subtraction	$\mathcal{O}(n^2)$	–	1

5. COMPARISON TO CB-DOA

A direct comparison between the beamspace and elementspace versions of CB-DoA algorithms is strongly dependent on two key factors. First, on the hardware platform in which the algorithms are tested, since any parallel hardware structure represents an advantage for the Beamspace CB-DoA. Second, on some previous knowledge about the source locations, since the processing of some of the sub-bands may be considered unnecessary.

If one sub-band of Beamspace CB-DoA is compared to the elementspace CB-DoA, the number of algebraic operations is larger for the beamspace version, but the operations are performed for matrices with smaller dimensions. For example, Beamspace CB-DoA requires 8 matrix multiplications, while the elementspace version requires only 3 matrix multiplications. On the other hand, there is a reduction on the number of entry operations in each matrix multiplication, due to the reduced dimensions of the structures, which may compensate the increase in the number of operations, since the complexity of multiplication is $\mathcal{O}(n^3)$.

6. SIMULATIONS

Some computer simulations were performed to assess the error performance of the Beamspace CB-DoA in comparison to the Beamspace ESPRIT. The metrics used for evaluation was the mean-squared error, defined as

$$\text{MSE} = \frac{1}{I}\sum_{i=0}^{I-1}|\theta_i - \hat{\theta}_i|^2, \tag{26}$$

where I denotes the number of Monte Carlo runs. In our simulation environment, $I = 250$. Simulations were run in a Matlab platform in a personal computer (PC). The transmitted sinusoids presented a time-varying amplitude, in order to simulate flat channel effects, according to a uniform distribution.

In the first simulation scenario, 2 transmitting sources and 18 receiving antennas were used in a noiseless environment, as well as in a scenario with SNR (signal-to-noise ratio) equals 25dB. The angles to be estimated were randomly chosen in the passband of the beamspace. The MSE was measured as a function of the number of beams, as shown in Fig. 5. The main goal of this scenario is to compare how the performance of both algorithms is affected by the number of beams used. As can be observed from Fig. 5, the MSE performance is equivalent for both algorithms in the noiseless and the 25dB scenarios. For 18 beams, Beamspace CB-DoA is equivalent to elementspace CB-DoA. Then, one may realize that MSE performance for Beamspace CB-DoA is inferiorly bounded both in the noiseless and in the $25-$ dB scenarios by the performance of its equivalent elementspace.

The MSE performances of both beamspace algorithms were also measured as function of the SNR, using 2 transmitting

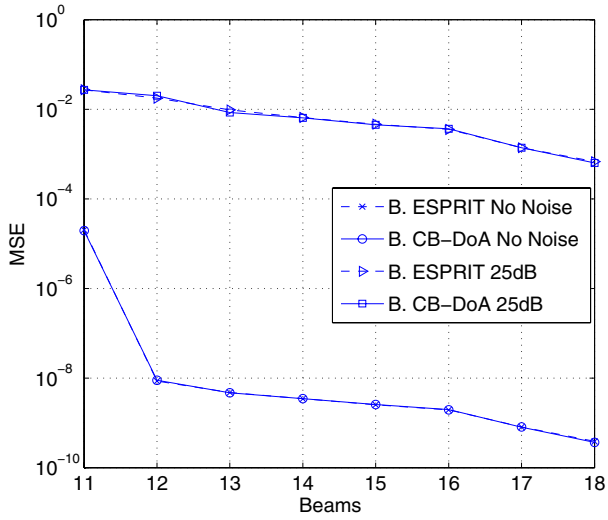

Fig. 5. Estimated MSE for Beamspace ESPRIT and Beamspace CB-DoA algorithms for 2 sources and 18 receiving antennas, as a function of the number of beams.

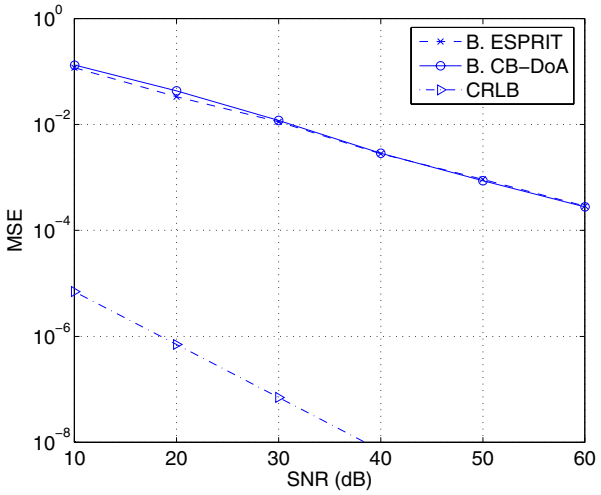

Fig. 6. Estimated MSE for Beamspace ESPRIT and Beamspace CB-DoA algorithms for 2 sources and 18 receiving antennas, as a function of the SNR, for 12 beams. The CRLB is also included.

antennas, 18 receiving antennas, and 12 beams in the beamspace. The MSE results are shown in Fig. 6. Once again, the performance of both algorithms are very similar for the whole range of SNRs simulated. In the same plot, the theoretical CRLB was included in order to assess the performance of both algorithms. The CRLB was calculated according to the simplified expression: [2]

$$\text{CRLB} = \frac{6\sigma_N^2}{N_s(N^2-1)N\sigma_s^2}, \tag{27}$$

where $N_s = 500$ represents the number of samples used in the simulation, and σ_s^2 represents the power of the sources. Such bound is an approximated underestimation in comparison to the actual CRLB, which does not cause much impact on the theoretical analyses, since it represents a bound from below to the variance of the error.

7. CONCLUSIONS

This article proposed an algorithm for DoA estimation, with lower computational complexity than the Beamspace ESPRIT [6]. The MSE performance of the proposed algorithm is similar to the one for the Beamspace ESPRIT, as confirmed through Monte Carlo simulations. In comparison to elementspace CB-DoA, there is a compromise between MSE performance and computational complexity. Moreover, the computational comparison to elementspace CB-DoA is dependent on the dimensions of beamspace transformation.

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