Low-Complexity DoA Estimation Based on Hermitian EVDs

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***Abstract*—The use of a receiving array of antennas allows the system to localize multiple sources. This estimation is sometimes simplified to a direction-of-arrival (DoA) estimation. The covariance-based (CB)-DoA algorithm is a lower complexity alternative to the ESPRIT algorithm, and yet achieves similar mean-squared error (MSE) with the same geometrical constraints. This article proposes a lower complexity CB-DoA algorithm. In the proposed algorithm, the eigenvector decomposition (EVD) of a non-Hermitian matrix is substituted by two lower complexity Hermitian EVDs. The replacement of EVDs does not imply MSE degradation. Discussions on exploiting the independent processing of the new EVDs are also provided.**

***Keywords*—direction-of-arrival estimation, ESPRIT, antenna array**

I. Introduction

The advantages of using multiple receiving antennas in wireless communications are well-known [1] [2]. The increase in the system capacity allows the support for a greater number of users. The use of an array of receiving antennas also provides the ability to spatially localize the sources, which is important for the design of beamforming filters [2].

In some applications, the exact localization of the source is not necessary. Then, the problem may be simplified to the discovery of the direction where the source lies. This is generally addressed as the direction-of-arrival (DoA) estimation problem.

The DoA problem was originally treated by either a maximum likelihood (ML) estimator or non-parametric wave estimators [2]. Early developments in the area led to the development of the Spectral MUSIC (multiple signal classification) [3] and the ESPRIT (estimation of parameters via rotational invariance techniques) [4] algorithms. Some algorithms related to Spectral MUSIC or ESPRIT became somewhat dominant in the 1-D (one dimensional) DoA estimation using linear arrays.

In the past two decades, research in DoA estimation became diverse and vast. The use of either planar or 3-D arrays has added new degrees of freedom in the design of algorithms [5]. The incorporation of iterative and adaptive approaches reduced the complexity of families of old algorithms, such as the classical ML [6]. The use of linear algebra tools also produced lower complexity algorithms by the use of reduced-rank techniques [7].

Some recent developments, for instance the CB-DoA [8] [9] algorithm, followed another path. CB-DoA is a lower complexity alternative to ESPRIT while requiring the same geometrical constraints.

This article proposes reducing even further the computational complexity of CB-DoA. The proposed algorithm replaces the EVD (eigenvector decomposition) of a non-Hermitian matrix by two Hermitian EVDs. Although the number of EVD operations increases, the overall computational complexity is reduced due to the comparatively lower cost of the Hermitian EVD. The proposed algorithm receives the denomination Hermitian-decomposition CB-DoA (HD-CB-DoA).

Section II describes the scenario and the basic mathematical modeling of the DoA estimation problem. Section III presents the proposed HD-CB-DoA algorithm. In section IV, the computational complexity of HD-CB-DoA is compared in relation to two benchmark algorithms, ESPRIT [4] and CB-DoA [8] [9]. Besides that, the advantages provided by the independence of the computation of the new EVDs are also discussed. Section V evaluates the mean-square error performance of the proposed as well as the two aforementioned benchmark algorithms. Section VI presents the conclusions of the article.

II. DoA Estimation

Consider a scenario where there are multiple sources transmitting narrowband signals, i.e. cisoids (complex sinusoids) with the same frequency. In the far field of the sources, there is a receiving linear array, which is co-planar to the sources. The propagation medium is assumed to be isotropic, without reflections and the transmitting antennas are omnidirectional on the plane of the receiving array. By using those hypotheses, the source localization is simplified to a direction finding problem, i.e. a direction-of-arrival (DoA) estimation [2]. For co-planar sources and linear receiving array, the one dimensional DoA is mapped into discovering the angle θ represented in Fig. 1.

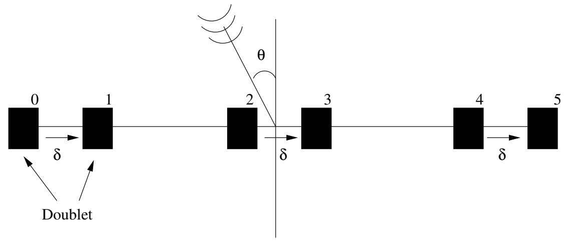

Fig. 1. Linear receiving array with 6 antennas. The geometry presents translational invariance constraints [4].

Consider a receiving linear array as depicted in Fig. 1. In order to apply DoA estimation algorithms based on rotational invariance subspaces, the receiving array must obey some geometrical constraints. It is required that the receiving array can be divided in two sub-arrays and there is a constant displacement vector $\boldsymbol{\delta}$ with initial point in antennas belonging to the first sub-array and the terminal point in antennas belonging to the second sub-array. In the array depicted in Fig. 1, the first sub-array contains the even-numbered antennas, and the second contains the odd-numbered antennas.

A. System Modeling

In the scenario described above, consider that there are M sources, each one transmitting cisoid $s_m(t), m \in \mathcal{Z}, 0 \leq m < M$. The mixture of signals propagates through the wireless medium. The signal wavefront reaches the N antennas of the receiving array under the DoA angle θ, as shown in Fig. 1.

Consider the signal acquired by the i^{th} antenna and sampled at $t = kT$, where k is a non-negative integer and T is the sampling period. Then using the discrete time k, one has that:

$$x_i(k) = \sum_{m=0}^{M-1} s_m(k - \tau_k(\theta_i)) a_i(\theta_m) + n_i(k), 0 \leq i < N \tag{1}$$

where $\tau_k(\theta_i)$ represents a propagation delay of the wavefront, θ_m is the DoA associated to the m^{th} source, $a_i(\theta_m)$ is the gain of the i^{th} antenna for the direction of the m^{th} source and $n_i(k)$ is the sampled AWGN (additive white Gaussian noise) acquired by the i^{th} antenna. Received signals $x_i(k), \quad i \in \mathcal{Z}, \quad 0 \leq i < N$ are supposed to be zero-mean signals.

The system may be represented by a vectorial modeling. Consider vector $\mathbf{x}(k)$ containing the N received, acquired and sampled signal at the N receiving antennas at discrete time k. Similarly, $\mathbf{n}(k)$ is defined by the sampled noise impinging on the N receiving antennas. Vector $\mathbf{s}(k)$ is defined as $\mathbf{s}(k) = [s_0(k - \tau_0(\theta_i)) \quad \dots \quad s_{M-1}(k - \tau_{M-1}(\theta_i))]^T$. Finally, $N \times M$-matrix $\mathbf{A}$ contains $a_i(\theta_m)$ at its i^{th} line and m^{th} column. The system is then modeled as:

$$\mathbf{x}(k) = \mathbf{A}\mathbf{s}(k) + \mathbf{n}(k). \tag{2}$$

B. Effect of the Geometrical Constraints

The geometrical constraints on the receiving array, represented in Fig. 1, generate a redundancy in the description of the received signals at both antennas of a doublet. Consider signals $x_0(k)$ and $x_1(k)$ acquired and sampled by antennas 0 and 1 in Fig. 1, respectively. There is a relation between $x_0(k)$ and $x_1(k)$ since the displacement vector $\boldsymbol{\delta}$ is known:

$$x_1(k) = \sum_{m=0}^{M-1} s_m(k - \tau_k(\theta_i)) a_1(\theta_m) + n_1(k)$$
$$= \sum_{m=0}^{M-1} s_m(k - \tau_k(\theta_i)) \mathrm{e}^{j\omega\delta \sin\theta/c} a_0(\theta_m) + n_1(k), \tag{3}$$

where $j = \sqrt{-1}$, $\omega = 2\pi f$ is the angular central frequency used and $\delta = \|\boldsymbol{\delta}\|_2$, for the constant displacement vector $\boldsymbol{\delta}$, shown in Fig. 1. The term inside the sum in Eq. (3) closely resembles the similar term in Eq. (1) with $i = 0$, except for a complex exponential, caused by a phase difference on the received wave on the antennas of the array. Consider $\mathbf{x}_a(k)$ as the sub-vector of $\mathbf{x}(k)$ containing the samples belonging to the antennas which are the initial points of the doublets, whereas $\mathbf{x}_b(k)$ is the sub-vector of $\mathbf{x}(k)$ contains samples from the terminal points of the doublets. Vector $\mathbf{x}_a(k)$ may be defined using a $P \times N, (N/2) \leq P < N$ selection matrix $\mathbf{J}_a$, which is a shortened $N \times N$ identity, containing the i^{th} line of the identity if and only if $x_i(k)$ is acquired by an antenna that is an initial point of a doublet. Then, $\mathbf{x}_a(k) = \mathbf{J}_a\mathbf{x}(k)$. Analogously, the selection matrix $\mathbf{J}_b$ is defined for data acquired by the terminal point of a doublet. Then, a new model is built for the system:

$$\mathbf{x}_a(k) = \mathbf{J}_a\mathbf{x}(k) = \mathbf{A}_a\mathbf{s}(k) + \mathbf{n}_a(k), \tag{4}$$
$$\mathbf{x}_b(k) = \mathbf{J}_b\mathbf{x}(k) = \mathbf{A}_a\boldsymbol{\Phi}\mathbf{s}(k) + \mathbf{n}_b(k), \tag{5}$$

where $\mathbf{A}_a = \mathbf{J}_a\mathbf{A}$, $\mathbf{A}_b = \mathbf{J}_b\mathbf{A}$, $\mathbf{n}_a(k) = \mathbf{J}_a\mathbf{n}(k)$ and $\mathbf{n}_b(k) = \mathbf{J}_b\mathbf{n}(k)$. Diagonal matrix $\boldsymbol{\Phi}$ presents the phase difference $\phi_m = \exp(j\omega\delta \sin\theta_m/c)$ at its m^{th} diagonal element. Then, in order to estimate the m^{th} DoA θ_m,

$$\hat{\theta}_m = \arcsin\left(\frac{c \ln(\phi_m)}{j\omega\delta}\right). \tag{6}$$

The CB-DoA algorithm [8] uses the EVD of the autocovariance $\mathbf{R}_{00} = E[\mathbf{x}_a(k)\mathbf{x}_a^H(k)]$, which is an Hermitian matrix, and the EVD of the crosscovariance $\mathbf{R}_{01} = E[\mathbf{x}_a(k)\mathbf{x}_b^H(k)]$, which is non-Hermitian. The EVD of non-Hermitian matrices presents higher computationally complexity than Hermitian EVDs [10]. In Section III, the proposed HD-CB-DoA is presented as an alternative, which uses only Hermitian EVDs.

III. Proposed Algorithm

The proposed HD-CB-DoA performs several of the operations conducted in the CB-DoA algorithm [8]. First, an EVD is executed on the Hermitian autocovariance matrix $\mathbf{R}_{00}$. The $P \times M$ matrix $\mathbf{U}_s$ containing the signal eigenvectors [4] in its columns and the $M \times M$ diagonal matrix $\boldsymbol{\Sigma}_s$ containing the square roots of the signal eigenvalues are generated.

Consider now the auxiliary matrix $\mathbf{V}$ given by

$$\mathbf{V} = (\boldsymbol{\Sigma}_s)^{-1}\mathbf{U}_s^H\mathbf{R}_{01}\mathbf{U}_s((\boldsymbol{\Sigma}_s)^{-1})^H. \tag{7}$$

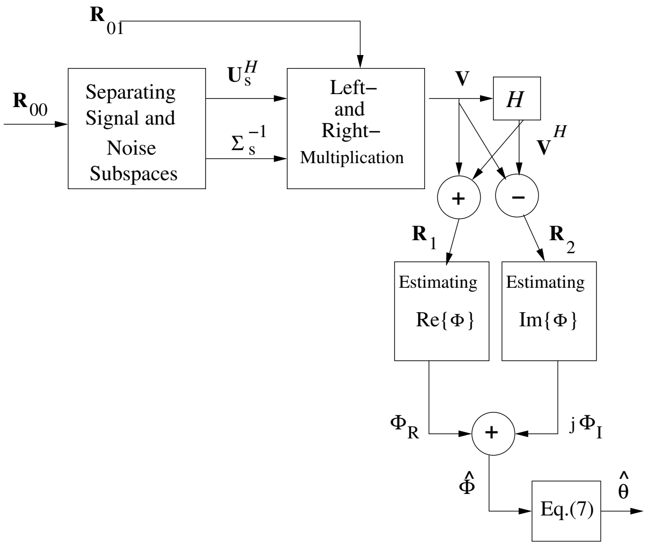

Fig. 2. Block diagram for representing operations of the proposed HD-CB-DoA algorithm. Independence of data verified on the vertical dataflux is discussed in Subsection IV-A.

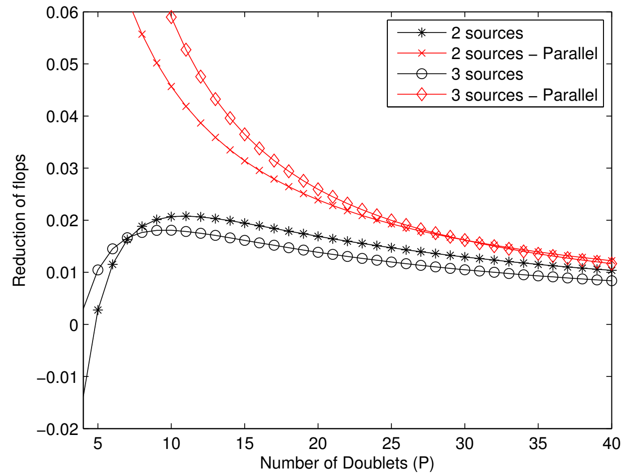

Fig. 3. Reduction R_{HDCB} in the amount of flops of HD-CB-DoA in comparison to CB-DoA for 2 and 3 sources (black curves), and $R_{\text{HDCB,par}}$ for the implementation suggested in Subsection IV-A (red).

In the CB-DoA algorithm [8], a non-Hermitian EVD is performed on $\mathbf{V}$. In the proposed HD-CB-DoA, two $M \times M$ Hermitian matrices $\mathbf{R}_1$ and $\mathbf{R}_2$ are generated:

$$\mathbf{R}_1 = \mathbf{V} + \mathbf{V}^H, \quad (8)$$
$$\mathbf{R}_2 = \mathbf{V} - \mathbf{V}^H. \quad (9)$$

Finally, lower-complexity Hermitian EVDs are applied on $\mathbf{R}_1$ and $\mathbf{R}_2$, whose eigenvalues estimates the real and imaginary parts of $\mathbf{\Phi}$, respectively: $\mathbf{\Phi}_R = \text{EVD}(\mathbf{R}_1)$ and $\mathbf{\Phi}_I = \text{EVD}(\mathbf{R}_2)$. Consequently, $\hat{\mathbf{\Phi}} = \mathbf{\Phi}_R + j\mathbf{\Phi}_I$. The estimate for the DoAs θ_m is obtained by Eq.(6).

A dataflow diagram representation for the whole HD-CB-DoA algorithm is found in Fig. 2.

IV. Computational Complexity Analysis

In Table IV, there is a comparison among the number of flops (Floating Point Operations) required by TLS-ESPRIT, CB-DoA and the proposed HD-CB-DoA. The definition of flop used here is the equivalent of a scalar complex multiplication or a scalar complex addition.

Since $P > M$, due to the constraints of the algorithms, the terms dependent on P are more critical to the overall computational complexity of the algorithms. The computational complexity of the operations are extracted from [10]. Whenever possible, we consider that the operations are performed by using tools in the Krylov space, such as in [11] and also described in [10]. The use of Krylov space tools was considered for the three algorithms on the EVDs, computed using Lanczos' iterations for the Hermitian case or Arnoldi's iterations for the non-Hermitian one [10].

The proposed HD-CB-DoA avoids the use of non-Hermitian EVDs, avoiding also the presence of $\mathcal{O}(n^3)$ terms. It is simple to notice in Table IV the advantages of complexity of both CB-DoA and HD-CB-DoA in comparison to ESPRIT. In order to compare the complexity of CB-DoA (denoted C_{CB}) and HD-CB-DoA (C_{HD}) in terms of flops, the reduction rate R_{HDCB} was used:

$$R_{\text{HDCB}} = \frac{C_{\text{CB}} - C_{\text{HD}}}{C_{\text{CB}}} \quad (10)$$

In Fig. 3, the results for metrics R_{HDCB} are shown for the scenarios with 2 and 3 sources. As shown in Fig. 3, since the metrics R_{HDCB} is positive for almost the whole range of calculated values for P, the complexity of HD-CB-DoA is lower than the complexity of CB-DoA by a ratio which varies between 0.01 and 0.02.

A. Further Reduction

Consider again Fig. 2. Some of the operations are independent, represented by the blocks on the vertical flux of the diagram. Therefore, they may be performed in parallel, which reduces the actual number of flops involved in serially computing the whole algorithm. There is a reduction of one subtraction of $M \times M$ matrices (see Section III) and one EVD of an $M \times M$ matrix (see Section III). Then, the expression for C_{HD} is simplified to:

$$C_{\text{HD,par}} = 2P^2 + P\log(P) + MP + 2M^2 + M\log(M) + 1.5M + M^2P + MP^2 \quad (11)$$

Similarly to Eq. (10), the reduction rate is defined

$$R_{\text{HDCB,par}} = \frac{C_{\text{CB}} - C_{\text{HD,par}}}{C_{\text{CB}}}, \quad (12)$$

which is evaluated in Fig. 3. For small values of P, there is a more significant reduction when the parallelization is taken into account. For two sources, $R_{\text{HDCB,par}}$ reaches 11% for $P = 3$. This is expected from Eq. (11), since the dropped terms in complexity are dependent on M. Whenever $P \gg M$, the terms dependent on P dominate and the advantages of parallelism are reduced.

TABLE I. COMPUTATIONAL COMPLEXITY COMPARISON ACCORDING TO THE VALUES LISTED IN [10] AND [11].

	Flops		
Operation (Complexity)	**TLS-ESPRIT**	**CB-DoA**	**Proposed**
Non-Hermitian EVD ($\mathcal{O}(4n^3/3+n^2)$)	$4M^3/3+M^2$	$4M^3/3+M^2$	–
Hermitian EVD ($\mathcal{O}(2n^2+n\log n)$)	$8P^2+2P\log(2P)+$ $8M^2+2M\log(2M)$	$2P^2+P\log P$	$2P^2+P\log(P)+$ $4M^2+2M\log(M)$
Multiplication ($\mathcal{O}(n^3)$)	$4M^2P+M^3$	M^2P+MP^2	M^2P+MP^2
Multipl. by Diagonal ($\mathcal{O}(n^2)$)	–	MP	MP
Add/Subtract ($\mathcal{O}(n)$)	–	P	$1.5M+M^2$
Full Inversion ($\mathcal{O}(2n^3/3)$)	$2M^3/3$	–	–
Diagonal Inversion ($\mathcal{O}(n)$)	–	P	M
Overall	$3M^3+9M^2+$ $8P^2+2P\log(2P)+$ $4M^2P+2M\log(2M)$	$4M^3/3+M^2+MP+$ $2P^2+P\log P+$ M^2P+MP^2+2P	$2P^2+P\log(P)+MP+$ $5M^2+2M\log(M)+1.5M+$ M^2P+MP^2
Dominant Terms	$\mathbf{8M^3+4M^2P+8P^2}$	$\mathbf{4M^3/3+M^2P+MP^2+2P^2}$	$\mathbf{M^2P+MP^2+2P^2}$

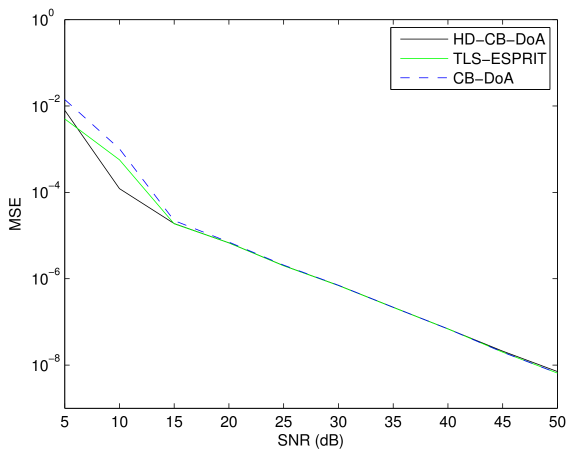

Fig. 4. MSE results for 2 sources and 8 doublets.

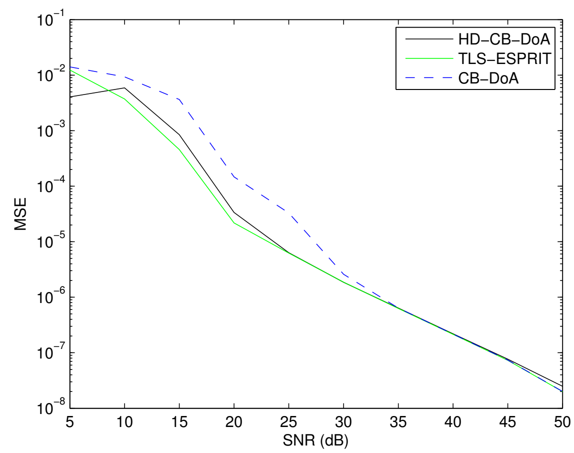

Fig. 5. MSE results for 3 sources and 12 doublets.

V. SIMULATION RESULTS

The proposed algorithm presents a lower computational complexity than its antecessors. In order to evaluate if it happens at the cost of degrading its error performance, Matlab simulations have been performed comparing HD-CB-DoA with CB-DoA and ESPRIT.

The simulated scenario comprises 2 sources transmitting at 1.5 GHz and $P = 8$ doublets in the reception. The receiving array presents an ULA (Uniformly Linear Array) geometry with a $\lambda/4$ space between two adjacent antennas. The transmitted symbols use a Gaussian random function modulating the cisoid at 1.5 GHz with zero mean and unit variance. The metrics used for the comparison is the Mean-Squared Error (MSE) given by:

$$\mathrm{MSE}(\hat{\boldsymbol{\theta}}) = \frac{1}{LT}\|\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}\|_2^2, \tag{13}$$

where $L = 300$ denotes the amount of Monte Carlo runs and $T = 10,000$ is the number of transmitted symbols per Monte Carlo run. The simulation results are presented in Fig. 4. The results for the 3 algorithms are similar for the whole range of the simulated SNRs. In this scenario, as presented in Section IV, the proposed HD-CB-DoA presents 1.8% less flops than CB-DoA and over 50% less flops than ESPRIT. When the independence of the new EVDs is explored, as presented in Subsection IV-A, the proposed HD-CB-DoA requires around 4.5% less flops to finish its execution in comparison to CB-DoA. In this case, the complexity reduction of HD-CB-DoA over ESPRIT exceeds 60%.

Fig. 5 shows the simulations for 3 sources and 12 doublets, with similar MSE results for the range of SNRs. HD-CB-DoA reduces around 1.6% the number of flops of CB-DoA and presents 47% of reduction over ESPRIT. When independence of EVDs is taken into account, proposed HD-CB-DoA needs 4.7% less flops than CB-DoA and around 50% less flops than ESPRIT.

VI. CONCLUSION

This article presents the novel HD-CB-DoA algorithm, which uses only Hermitian matrices in its eigendecompositions. The proposed algorithm presents less computational complexity, with similar MSE results in comparison to both ESPRIT and CB-DoA. When the independence of the new EVDs is explored, the reduction in complexity provided by HD-CB-DoA is more significant. Possible continuations for this work include incorporating transformation to a real subspace [12].

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